Linking in the 3-body problem

The Centennial of the International Mathematical Union
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Étienne Ghys, CNRS, UMPA-ENS Lyon, Académie des sciences

Linking numbers

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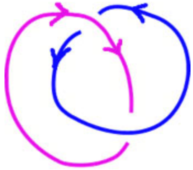

Linking $= +1$

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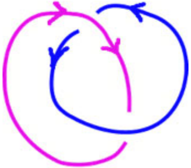
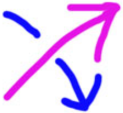


Linking $= +1$

-1



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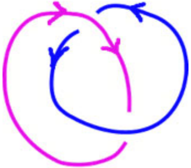

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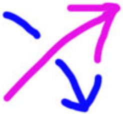
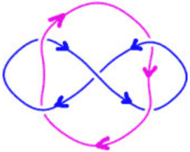
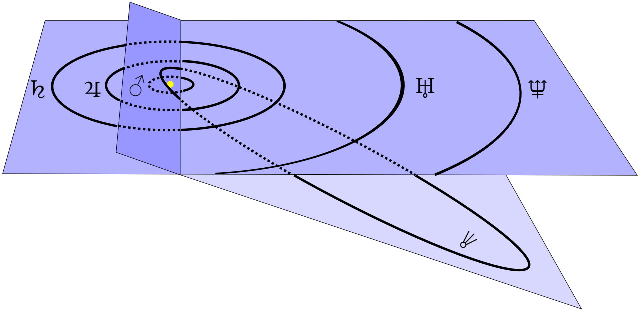


$+1.$

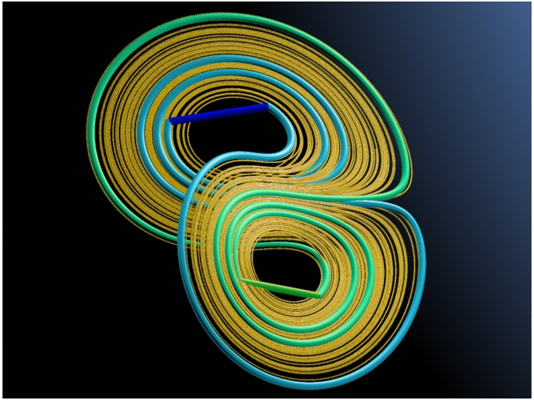


Linking $= 0$

Gauss



Left handed vector fields



X : a vector field on the 3-sphere, generating a dynamical system ϕ_t.

$$k(x, T) = [x \xrightarrow{\phi} \phi_T(x) \xrightarrow{segment} x]$$

$$\textit{linking}\,(k(x_1, T_1), k(x_2, T_2))$$

Definition : *A vector field X on the 3-sphere, generating a dynamical system ϕ_t, is called left handed if for any two distinct points x_1, x_2 the trajectories starting from x_1 and x_2 link positively :*

$$\liminf_{T_1, T_2 \to \infty} \frac{1}{T_1 T_2} \mathit{linking}\, (k(x_1, T_1), k(x_2, T_2)) > 0.$$

Stable property : Small C^1 perturbations of left handed vector fields are left handed.

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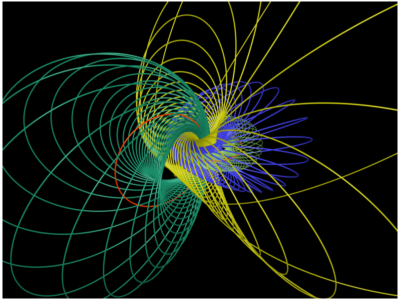
Example

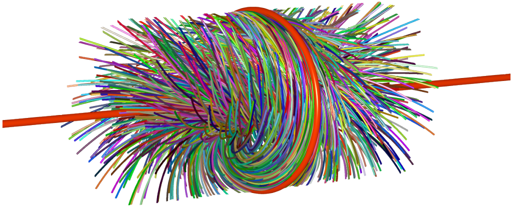
Two independent oscillators : dynamics on $\mathbf{C}^2 \simeq \mathbf{R}^4$:

$$\phi^t(z_1, z_2) = (\exp(it)z_1, \exp(it)z_2)$$

- $|z_1|^2 + |z_2|^2$ is invariant, so the dynamics is on spheres $\mathbf{S}^3$ in $\mathbf{R}^4$.
- Orbits are the fibers of the Hopf fibration :

$$(z_1, z_2) \in \mathbf{S}^3 \subset \mathbf{C}^2 \mapsto \frac{z_1}{z_2} \in \mathbf{C} \cup \{\infty\} \simeq \mathbf{S}^2.$$





Characterization

Let $\mathcal{P}$ be the compact convex set of probability measures which are invariant under the flow (for instance, periodic orbits).

There is a well defined quadratic linking form :

$$\mathit{link} : (\mu_1, \mu_2) \in \mathcal{P} \times \mathcal{P} \mapsto \mathit{linking}(\mu_1, \mu_2) \in \mathbf{R}$$

Theorem : *A vector field X on the 3-sphere is left handed if and only if the linking quadratic form is positive.*

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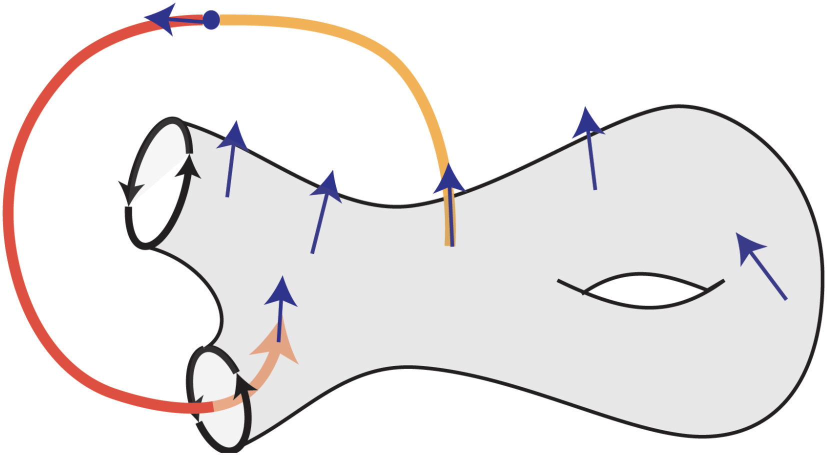
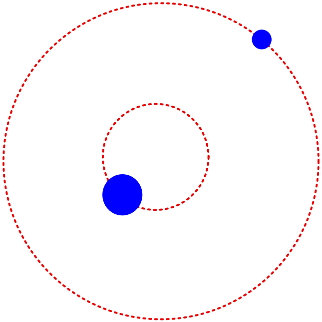
Any finite collection of periodic orbits is the binding of some Birkhoff section, an open book transverse to the vector field.



A conjecture

Restricted planar circular 3 body problem



A conjecture

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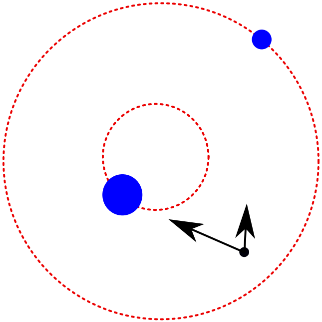


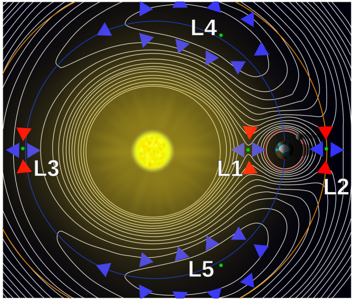
Conjecture

The Restricted Circular Planar 3 body Problem is left handed up to the first Lagrange point.

3 body in a rotating frame

$$\frac{1}{2}(u^2 + v^2) - V(x, y) = Constant = h$$

$$V(x, y) = \frac{1-\mu}{\sqrt{(x+\mu)^2 + y^2}} + \frac{\mu}{\sqrt{(x+\mu-1)^2 + y^2}} + \frac{1}{2}(x^2 + y^2)$$



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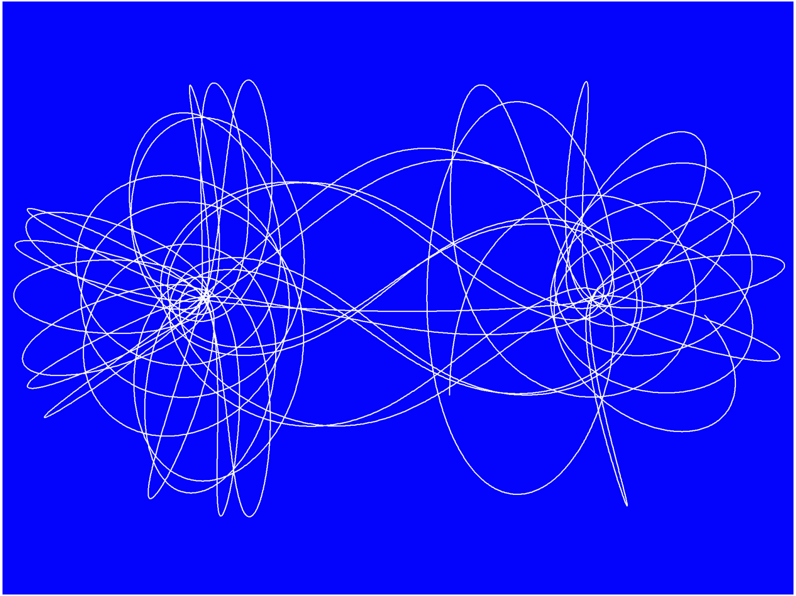
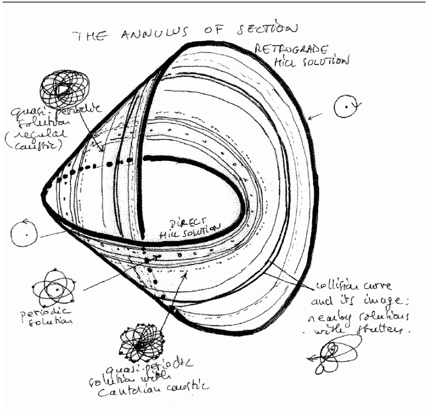


If the energy is $<$ than the energy of the first Lagrange point, the configuration space, after desingularization, is a 3-sphere.

If the energy is $<< 0$ there is a Birkhoff section (Poincaré).
Birkhoff's conjecture 1915



Geodesics on a convex surface

SUR LES LIGNES GÉODÉSIQUES DES SURFACES CONVEXES*

PAR

HENRI POINCARÉ

M. HADAMARD l'a bien compris, et c'est ce qui l'a déterminé à étudier les lignes géodésiques des surfaces à courbures opposées; il a donné une solution complète de ce problème dans un memoire du plus haut intérêt. Mais ce n'est pas aux géodésiques des surfaces à courbures opposées que les trajectoires du problème des trois corps sont comparables, c'est au contraire aux géodésiques des surfaces convexes.

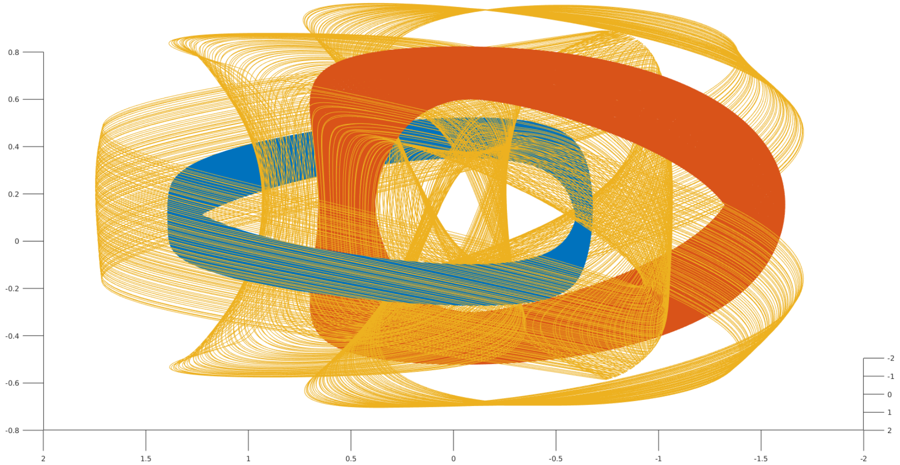
J'ai donc abordé l'étude des lignes géodésiques des surfaces convexes; malheureusement le problème est beaucoup plus difficile que celui qui a été résolu par M. HADAMARD. J'ai donc dû me borner à quelques resultats partiels, relatifs surtout aux géodésiques fermées qui jouent ici le rôle des solutions périodiques du problème des trois corps.

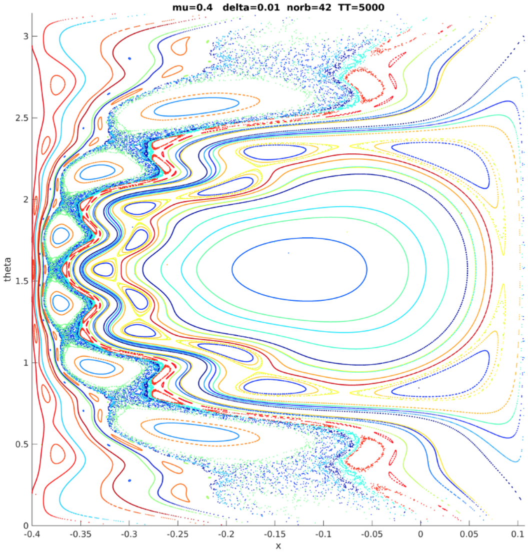
Theorem : Anna Florio and Umberto Hryniewicz, 2021

The geodesic flow on a convex surface is left handed if the curvature is 0.7225 *pinched.*

Three body problem

Marie Lhuissier




mu=0.4 delta=0.01 norb=42 TT=5000
theta
x
3
2.5
2
1.5
1
0.5
0
-0.4
-0.35
-0.3
-0.25
-0.2
-0.15
-0.1
-0.05
0
0.05
0.1

The PR3BP seems to be left handed, at least up to the first Lagrange point (Marie Lhuissier).

```
load('atable.mat','T1')
T1
% mu : parametre de masse   ( corps de masses 1-mu,mu en -mu,-mu+1)
% delta : ecart a l'energie du point de Lagrange L1
% enl(min,max,moy) : enlacement(min,max,moy) normalise par unite de temps^2
% calcule sur nbenl couples de trajectoires
% tenl : taux d'enlacement sur le niveau d'energie (invariant d'Arnold
% normalise par le volume au carr) calcule par la formule integrale


T1 =

  708 table
```

mu	delta	E0	enlmin	enlmax	enlmoy	tenl	nbenl
0.1	0.001	-1.7995	0.14122	1.0831	0.40445	0.38961	866
0.1	0.01	-1.8085	0.14658	1.0911	0.34332	0.40153	892
0.1	0.1	-1.8985	0.20524	1.217	0.51279	0.51388	948
0.1	0.15	-1.9485	0.24948	1.3026	0.53924	0.57872	625
0.1	0.2	-1.9985	0.29186	1.4666	0.62804	0.64643	957
0.1	0.25	-2.0485	0.36077	1.5293	0.7076	0.71726	541
0.1	0.3	-2.0985	0.40807	1.7233	0.87044	0.79141	588
0.1	0.35	-2.1485	0.47224	1.7325	0.89485	0.86914	568
0.1	0.4	-2.1985	0.49441	1.8661	0.954	0.9505	549
0.1	0.45	-2.2485	0.56716	1.9891	1.0836	1.0356	568
0.1	0.5	-2.2985	0.050644	2.1112	0.96931	1.1247	976
0.1	1	-2.7985	1.489	3.5378	2.3216	2.2535	533
0.2	0.001	-1.9033	0.22144	1.2569	0.46926	0.52154	685
0.2	0.01	-1.9123	0.24165	1.3658	0.54993	0.53676	704
0.2	0.05	-1.9523	0.26932	0.70626	0.423	0.59925	171
0.2	0.1	-2.0023	0.31328	1.392	0.63835	0.67747	857
0.2	0.15	-2.0523	0.36021	1.5941	0.71101	0.75836	786
0.2	0.2	-2.1023	0.41736	1.6976	0.81927	0.84276	667
0.2	0.25	-2.1523	0.51341	1.7925	0.90636	0.93118	694
0.2	0.3	-2.2023	0.55281	1.9485	0.9826	1.0238	515
0.2	0.35	-2.2523	0.63514	2.1411	1.1952	1.1208	351
0.2	0.4	-2.3023	0.72592	2.154	1.2596	1.2225	406
0.2	0.45	-2.3523	0.89235	2.3342	1.4531	1.3291	378
0.2	0.5	-2.4023	0.85778	2.5035	1.3684	1.4405	722
0.2	1	-2.9023	1.9829	4.0518	2.8353	2.8553	532
0.3	0.001	-1.9611	0.31059	1.5285	0.68452	0.62269	463
0.3	0.01	-1.9701	0.29872	1.4582	0.6008	0.64139	478
0.3	0.1	-2.0601	0.4067	1.5997	0.82247	0.81195	459
0.3	0.15	-2.1101	0.47069	1.7421	0.89633	0.90987	435