Geometrization

Gang Tian

Peking University

In last twenty years, geometrization has played a profound role in applications of geometric methods to studying problems which arise from other fields, e.g., topology, algebraic geometry.

Let M be a compact manifold of dimension n, by a geometric structure, we mean a Riemannian metric g, locally, it is a positive definite matrix-valued function $(g_{ij})_{1\leq i,j\leq n}$.

There are too many possibilities of geometric structures on M, we need to find one which is useful.

The chosen one is given by prescribing certain properties on its curvature. Recall that curvature $R(g)$ of a metric g is invariant under diffeomorphisms and measures how the space (M, g) is curved.

Let us start with surfaces. i.e., $n = 2$:

Now curvature is determined by a function K, referred as the Gauss curvature. The most natural condition on a metric is to make K constant. How?

Start with any initial metric g_0, one may try to find u such that $g = e^u g_0$ has constant $K(g)$.

This is the geometrization in dimension 2.

The approach by complex analysis:

g_0 induces a conformal structure on M to make it become a Riemann surface. Then, by the Riemann mapping theorem, the universal covering of M is either S^2, or $\mathbb{C}$, or H^2.

It follows that there is a metric of the form $g = e^u g_0$ with constant Gauss curvature.

The approach by PDE:

If $g = e^u g_0$ has constant Gauss curvature K, then u satisfies

$$\Delta u + K(g_0) \;=\; Ke^u.$$

It can be solved by elliptic method (Nirenberg, Kazdan-Warner, Aubin etc.) and late, also solved by Ricci flow (R. Hamilton, B. Chow etc.).

Once we can solve the equation, we know that the universal covering of M is either S^2, or $\mathbb{C}$, or H^2.

This approach is preferred to achieving the geometrization.

What about higher dimensions?

Definition A metric g is Einstein if it satisfies the Einstein equation:

$$\mathrm{Ric}(g) = \lambda g,$$

where λ is a constant and $\mathrm{Ric}(g)$ is the Ricci curvature which measures deviation of volume form from the Euclidean one.

The equation on $\mathrm{Ric}(g)$ is a non-linear system and very difficult to solve.

Now assume that M is a compact 3-manifold. If M admits an Einstein metric, then its universal covering is either S^3 or $\mathbb{R}^3$ or H^3.

However, in general, not every M admits an Einstein metric.

Thurston's geometrization conjecture states that any 3-manifold can be decomposed in a natural way into these Einstein ones plus some Graph manifolds.

The Poincare conjecture is the special case in which M is simply connected.

Perelman solved Thurston's conjecture by using the Ricci flow introduced by Hamilton:

$$\begin{cases} \dfrac{\partial g}{\partial t} = -2\mathrm{Ric}(g) \\ g(0) = \text{a given metric} \end{cases}$$

If this flow has a global solution $g(t)$, possibly after appropriate scaling, which converges to a smooth metric g_∞ as $t \to \infty$, then g_∞ is Einstein, so the universal covering of M is standard.

However, Ricci flow develops singularity. Perelman established a topological classification of finite-time singularity. This is crucial in his solution.

Geometrization of 3-manifolds: Let M be any closed 3-manifold with an initial metric g_0, we have

1. There is a unique Ricci flow with surgery $g(t)$ with $g(0) = g_0$;

2. $g(t)$ becomes a classical solution when $t > t_0$ for a large t_0.

3. Either $g(t)$ becomes extinct at finite time or after an appropriate scaling, $g(t)$ converge to a finite set of canonical metrics $(M_i, g_{\infty,i})$.

This is a refined version of Thurston's geometrization conjecture. Bamler has essentially completed part 2 and 3.

More recently, Bamler-Kleiner had found a way of constructing Ricci flow with surgery which depends on initial values continuously. This solves part 1 in some weaker sense.

As an application, they solved the generalized Smale conjecture: The diffeomorphism group of every 3-dimensional spherical space form deformation retracts to its isometry group.

A solution for the original Smale conjecture for S^3 was previously given by Hatcher.

In dimension 4, Ricci flow provides a promising approach to proving the 11/8-conjecture proposed by Matsumoto: For any smooth spin 4-manifold, the ratio of its 2nd Betti number and signature is least $11/8$.

To complete this approach, one needs to have a sufficient understanding of how Ricci flow evolves. Recently, Bamler developed a regularity theory on Ricci flow in higher dimensions. He has obtained a number of surprising results on Ricci flow, especially, in dimension 4.

In dimension 4, there are also anti-self-dual metrics, i.e., those with vanishing half Weyl tensor: $W_+ = 0$.

If M is homotopic to the 4-sphere and admits an anti-self-dual metric, then it is diffeomorphic to the 4-sphere.

There are many examples of anti-self-dual metrics, but it is still missing a method of deforming a given metric towards anti-self-dual metric, possibly though finitely many surgeries.

Another generalization of 2-dimensional geometrization is for compact Kähler manifolds.

Now we assume M is a complex manifold and g is given locally by Hermitian positive matrices $(g_{i\bar{j}})$ satisfying:

$$g_{i\bar{j}} = \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}.$$

A nice fact is that the Ricci flow preserves the Kählerian condition.

What is special about Kähler manifolds?

• Ricci flow can be reduced to a flow on scalar functions.

As an example, we recall a result of Z. Zhang and myself:

There is a unique maximal solution $g(t)$ on $[0, T)$, where T is explicitly determined by the Kähler class $[g(0)]$ and the first Chern class $c_1(M)$, more precisely,

$$T = \sup\{t \,|\, [g(0)] - t\, c_1(M) > 0\}.$$

In particular, if $c_1(M) \leq 0$, the Ricci flow has a global solution.

The Analytic Minimal Model Program, initiated by Song and myself, is to classify compact Kähler manifolds birationally through Ricci flow:

1. Construct a solution with surgery for Ricci flow, like what we did for 3-manifolds;

2. Prove the solution above either becomes extinct in finite time or after normalization, converges to a twisted Kähler-Einstein metric.

Many progress has been made. The main obstacle is to understand finite-time singularities. We expect that these singularities correspond to flips in algebraic geometry.

There are other geometrization problems, which are less understood than previous cases.

The first is provided by the Hermitian curvature flow of J. Streets and myself:

$$\frac{\partial g}{\partial t} = -S + Q, \quad g(0) = g_0,$$

where g are Hermian metrics, S is the “Ricci” curvature of the Chern connection and Q is a quadratic function on torsion.

The Hermitian curvature flow preserves the pluri-closed condition as well as the generalized Kähler condition for Hermitian metrics. The success of the geometrization through this flow will enable us, among other things, to classify Class VII surfaces.

Finally, we mention the symplectic flow introduced by J. Streets and myself. We hope that this flow will be useful in studying symplectic manifolds.

Thanks!