Rigid Local Systems: Arithmetic Properties

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Rigid Local Systems (RLS)

- X topological space, for us *complex algebraic variety*;
- irreducible *local system* (LS) $\stackrel{\text{dfn}}{=}$ irreducible $\rho : \pi_1(X, x) \to GL_r(\mathbb{C})$ up to gauge transformation;
- i.e.: irreducible fibre bundle $\mathcal{V}_\rho \to X$ with locally constant transition functions.

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- ρ *rigid*: deformation $\rho_t : \pi_1(X, x) \to GL_r(\mathbb{C}[[t]])$ gauge equivalent to ρ_0, i.e. $\exists g_t \in GL_r(\mathbb{C}[[t]])$ with $\rho_t = g_t \rho_0 g_t^{-1}$, i.e. moduli point $[\mathcal{V}_\rho]$ isolated.
- May fix determinant and conjugacy classes of local monodromy at ∞ for X not proper.

Occurence

dim $(X) = 1$ Katz: $\exists$ RLS $\Longrightarrow$
• $X = \mathbb{P}^1 \setminus \{\text{finitely many points}\}$;
• moduli points have no multiplicity;
• all RLS *come from geometry*, i.e. 'like' (summand of)
$\mathcal{V}_{\rho,\tau\in\mathbb{P}^1\setminus\{\infty,\tau_1,\ldots,\tau_n\}} = H^1_c(Y_\tau)$, $Y_\tau \subset \mathbb{A}^2 : \; y^N = \prod_1^n (x-\tau)(x-\tau_i)$.

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Shimura varieties of rank ≥ 2: Margulis superrigidity $\Longrightarrow$
- all irreducible LS are rigid;
- moduli points have no multiplicity;
- **but** we do not know whether they come from geometry.

Simpson's Geometricity Conjecture

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Petrov's theorem (2020) inserts it in a vast arithmetic program

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Geometricity conjecture inacessible, except in dim. 1 (Katz).
Instead study consequences.

Consequences of Simpson's geometricity conjecture

Integrality conjecture (Simpson 1990)

RLS are integral, i.e. $\rho : \pi_1(X, x) \to GL_r(\mathcal{O})$, $\mathcal{O}$ number ring.
$\ll$ Betti cohomology is defined over $\mathbb{Z}$. $\gg$

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Crystalline conjecture (E-Groechenig 2018)

connection corresponding via the Riemann-Hilbert correspondence to a RLS $/X_{\mathbb{Q}_q}$ for a.a. p is
i) an isocrystal with a Frobenius structure;
ii) underlying a p-adic crystalline representation (Fontaine).
$\ll$ Gauß-Manin connections are so by Deligne. $\gg$

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Remark

- Not a single RLS known with moduli point with higher multiplicity.
- Crystallinity applies for all proper or rank ≥ 2 Shimura varieties.

Methods

Integrality

uses Langlands program (Deligne's companions' conjecture, proven by L. Lafforgue and Drinfeld).

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Crystallinity

uses theory of Higgs-de Rham flows (Lan-Sheng-Zuo).

One consequence of our crystallinity theorem

Pila-Ananth Shankar-Tsimerman (posted on Sept. 21 2021)

André-Oort conjecture for all Shimura varieties.
[i.e: Zariski closure of special points is special $\subset$ Shimura variety.]

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