Coulomb Branches of 3d Supersymmetric Gauge Theories

construction of new varieties

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Mathematics without Borders: The Centennial of the IMU

——— joint works with Braverman, Finkelberg

Motivated by theoretical physics — Seiberg, Witten, Hanany
Kapustin, ...
mid 90's ~

Mathematical Approach — More recent 2015 ~

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A construction of varieties by a new method

— a technique from geometric representation theory

— an idea from topological quantum field theory
TQFT

Our Construction uses the language of schemes.

$X = \mathbb{C}^n$ vs $\mathbb{C}[x_1, \cdots, x_n]$ polynomial ring

ring of functions on X

set of maximal ideals

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an affine scheme $X \rightleftarrows$ a commutative ring $\mathbb{C}[X]$

gluing $\to$ a scheme

<u>Example</u> "quotient" $X/\!/G \rightleftarrows \mathbb{C}[X]^G$ ring of invariants

$\longrightarrow$ geometric invariant theory

a method to construct new varieties

Coulomb branch $\longleftarrow$ commutative ring

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Prototype of the construction : group ring $R[G]$ of a finite group G

R : commutative ring

$R[G] = \{ \varphi : G \longrightarrow R \}$

$$(\varphi * \psi)(x) \overset{\text{def.}}{=} \sum_{y \in G} \varphi(y^{-1}x)\psi(y)$$ convolution product

<u>Remark</u> : $R[G]$ is commutative $\Longleftrightarrow$ G : abelian

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We use the same idea for $G \rightsquigarrow$ a certain topological space

$R[G] \rightsquigarrow$ its homology group

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2-manifold $\Sigma^2 \rightsquigarrow$ vector space $Z(\Sigma^2)$

3-manifold M with bdry ∂M $\mapsto Z(M) \in Z(\partial M)$

+ Composition of cobordisms $\rightsquigarrow$ composition of linear maps etc.

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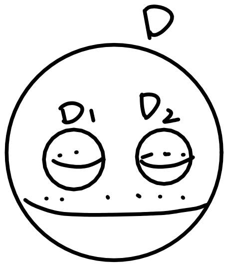
Then 

$D \setminus D_1 \cup D_2 \rightsquigarrow Z(S^2) \otimes Z(S^2) \longrightarrow Z(S^2)$

commutative multiplication

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Open Problem Hyperkähler metrics on Coulomb branches?

(more interesting for 4d gauge theories)