Of crystals and corals

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based on joint work with

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Skoltech
Skolkovo Institute of Science and Technology

Crystals and corals

Manganese oxide dendrites on a limestone bedding plane

Hele-Shaw cell, by Yulia Petrova

Electrodeposition in a copper sulfate solution by Kevin R Johnson

Mold and lightning

Mold growth - gift by Sir Alexander Fleming to Edinburgh University Library

Lichtenberg figure

Two models: DLA and DBM

FIG. 1. Random aggregate of 3600 particles on a square lattice.

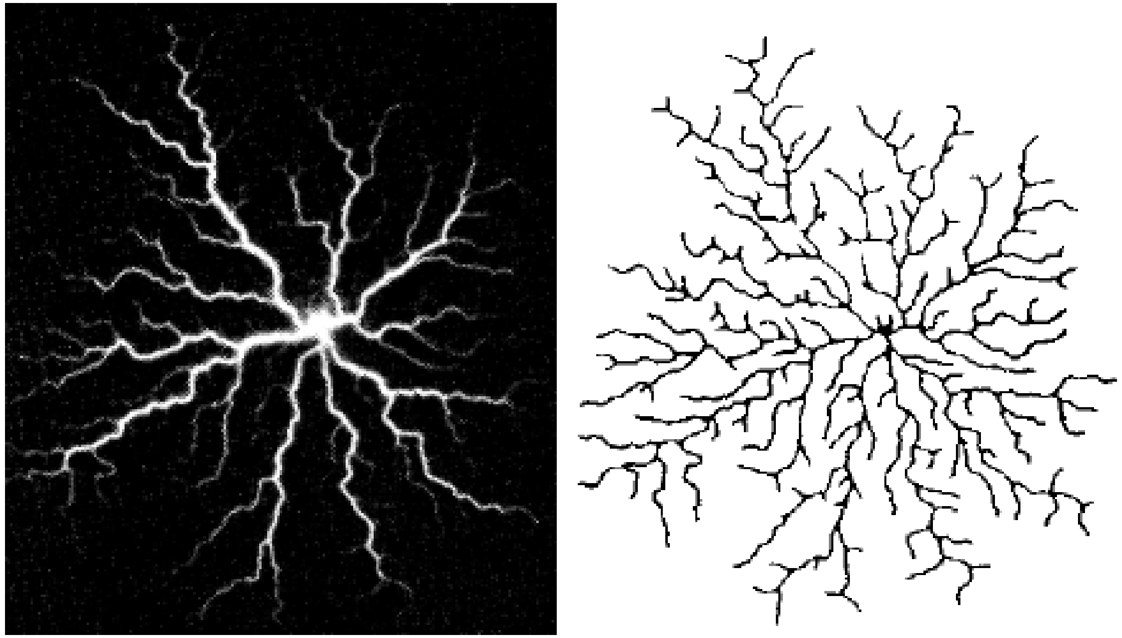
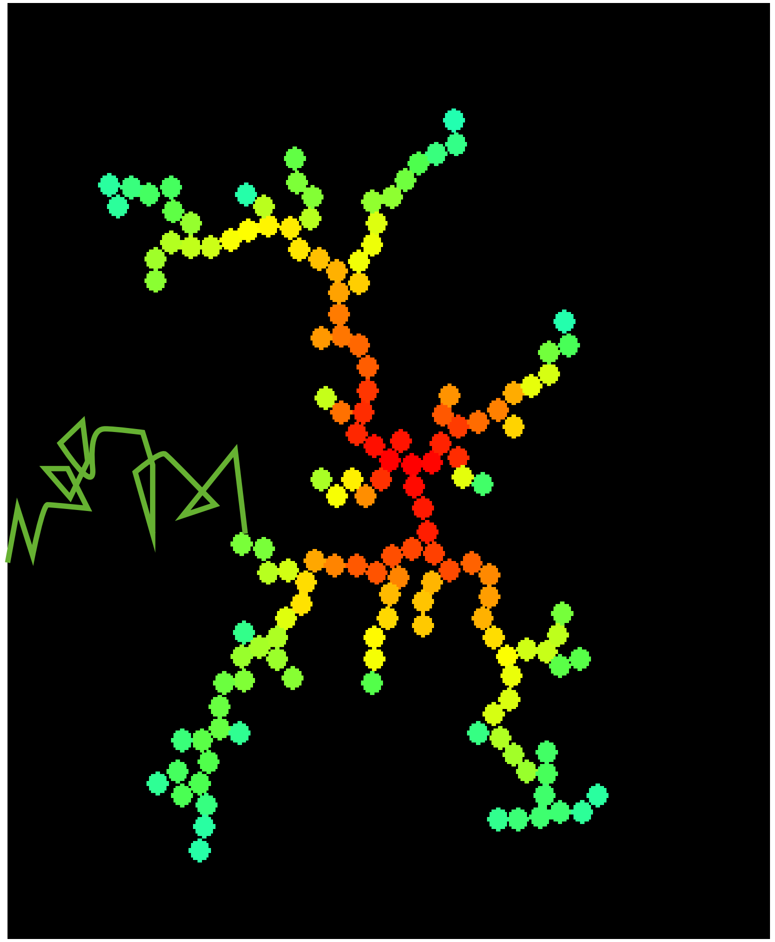


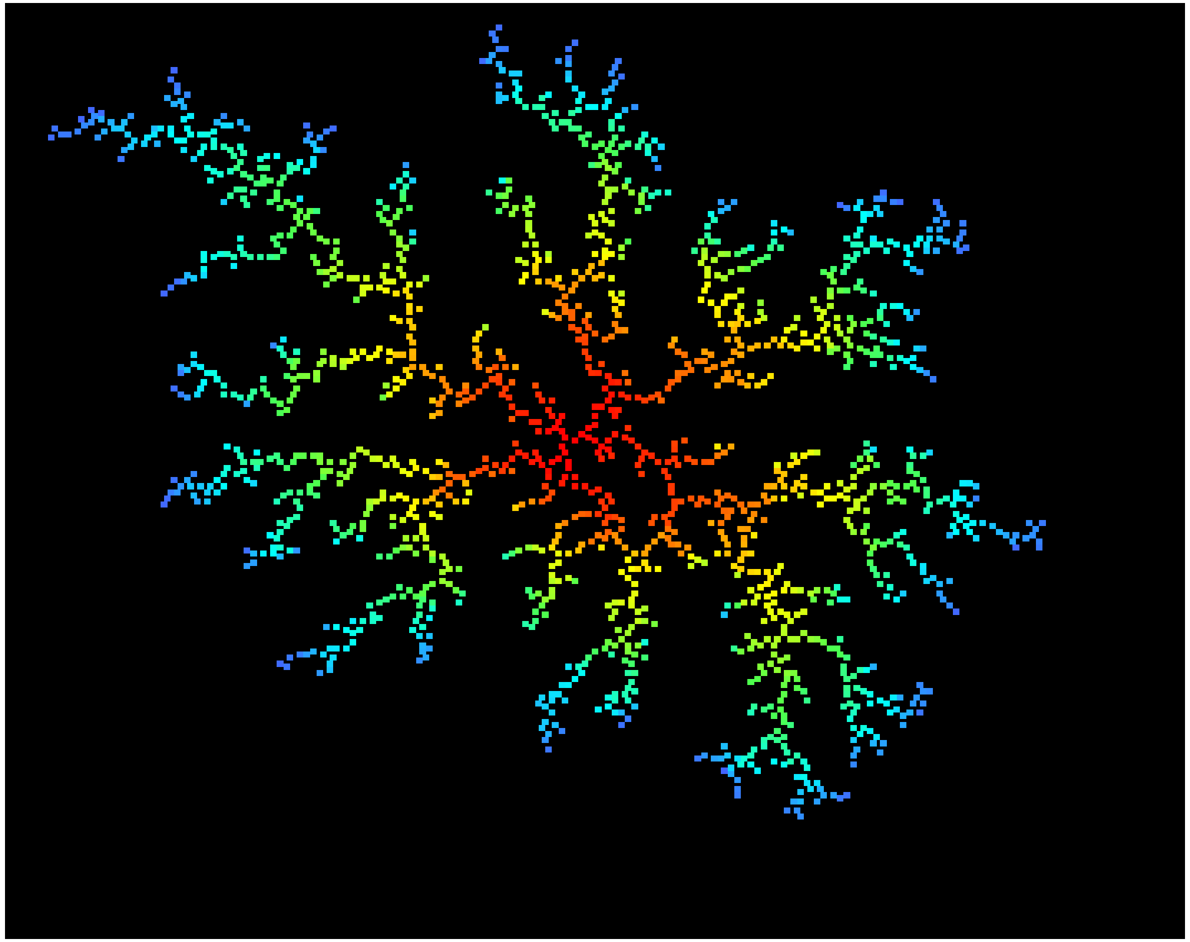
FIG. 1. Time-integrated photograph of a surface leader discharge (Lichtenberg figure) on a 2-mm glass

Diffusion Limited Aggregation (DLA) was introduced by **T.A. Witten and L.M. Sander** in 1981 to model a number of physical phenomena.

Dielectric Breakdown Model (DBM) is a one parameter generalization, introduced by **L. Niemeyer, L. Pietronero, and H. J. Wiesmann** in 1984

Diffusion Limited Aggregation





Send particles from far away by Random Walk (or Brownian Motion), until they touch the cluster and become attached. The probability to attach is proportional to **harmonic measure**

Harmonic measure

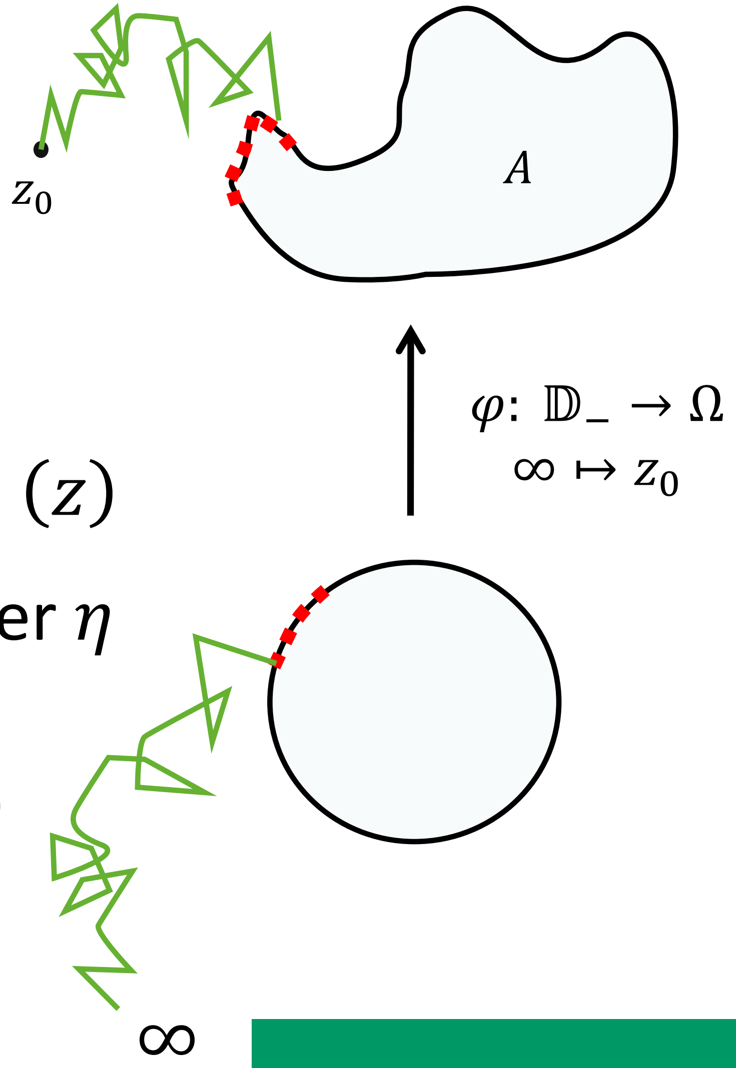
DLA: probability to attach a particle is given by harmonic measure

Outside $\Omega = \hat{\mathbb{C}} \setminus A$ of a cluster A , viewed from z_0 (we take $z_0 = \infty$)

ω definitions

- **Brownian motion:** hitting probability
- **conformal map:** image of the length
- **potential theory:** equilibrium measure,

  ω_∞ minimizes $\iint -\log|u - v| \, d\omega(u) \, d\omega(v)$
- **Dirichlet:** if $\Delta u = 0$ then $u(z_0) = \int_A u(z) d\omega_{\Omega, z_0}(z)$



DBM: probability to attach is proportional to a power η of harmonic measure (or Green's function):

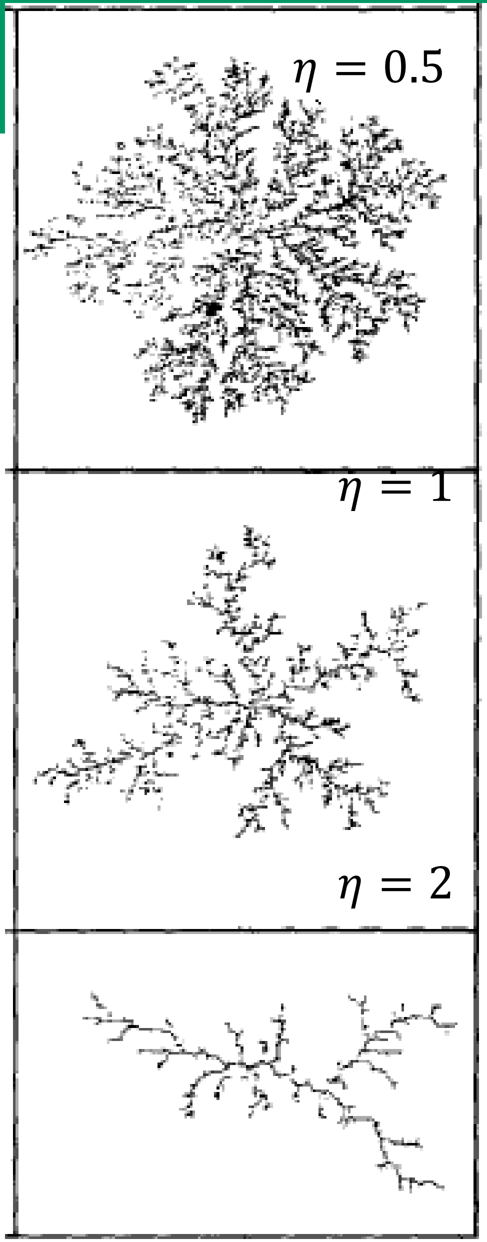
$\mathbb{P}_j = \boldsymbol{\omega}_j^{\eta} / \sum \boldsymbol{\omega}_j^{\eta}$ (for DLA $\mathbb{P}_j = \omega_j = \omega_j / \sum_k \omega_k$)

Bigger powers favor «spikes»

$\eta = 1$ is DLA, while $\eta = 0$ is M. Eden model

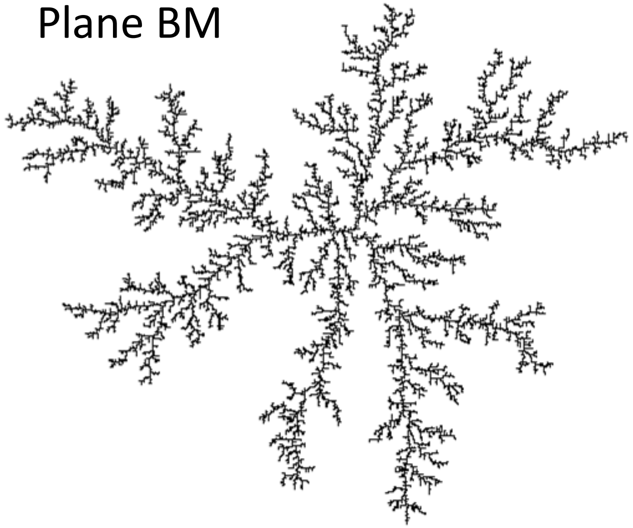
Diffusion Limited Aggregation

Delectric Breakdown Model

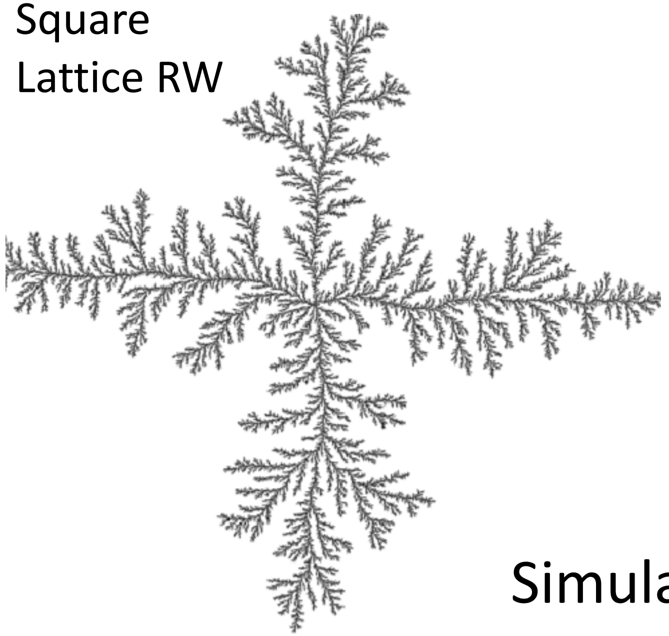


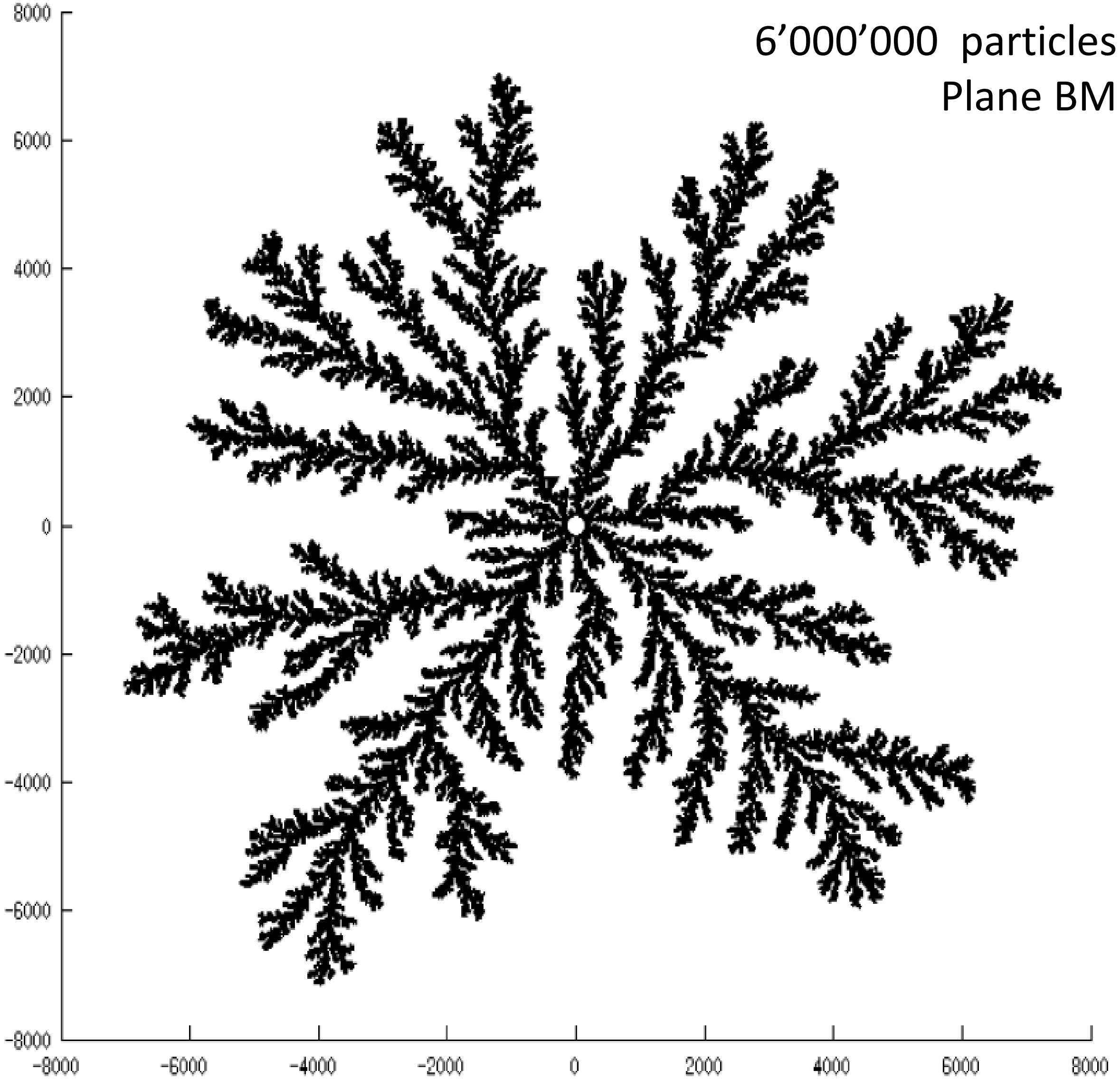
By Niemeyer et al

Plane BM



Square
Lattice RW





Simulations by Dmitry Beliaev and Denis Grebenkov

ω : fjords and spikes

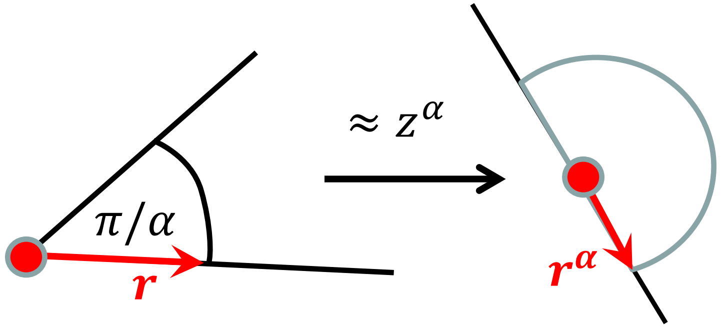
Scaling of ω near corners:

$$\omega B_r \approx (r/R)^{\alpha}$$

for an r-ball in a set of radius R.

Clearly $\frac{1}{2} \leq \alpha \leq \infty$

Angle	α
2π	$1/2$
π	1
$\pi/2$	2
0	∞



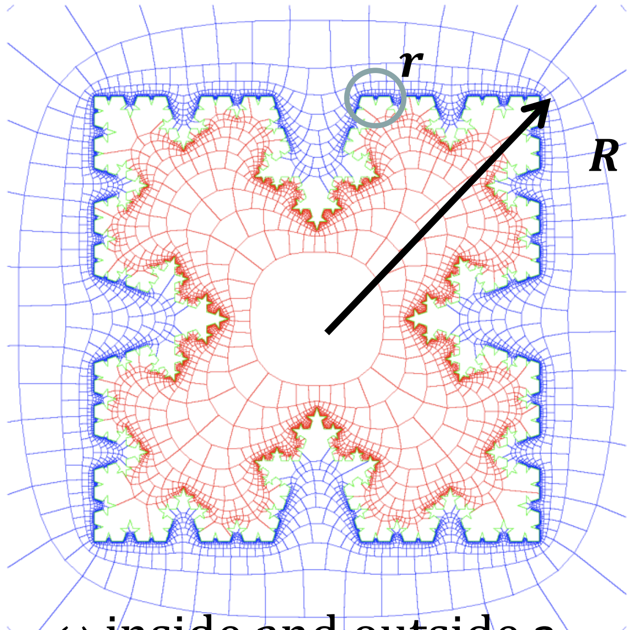
Beurling's estimate: for any ball

$$\omega(B_r) \lesssim \sqrt{r/R}$$

Hence, for an individual site j, $\omega_j \lesssim 1/\sqrt{R}$

Makarov's theorem:

Harmonic measure has Borel support of **dimension 1**. In discrete setting (cf Lawler) ***most of ω lives on sites with $\omega_j \asymp 1/R$***

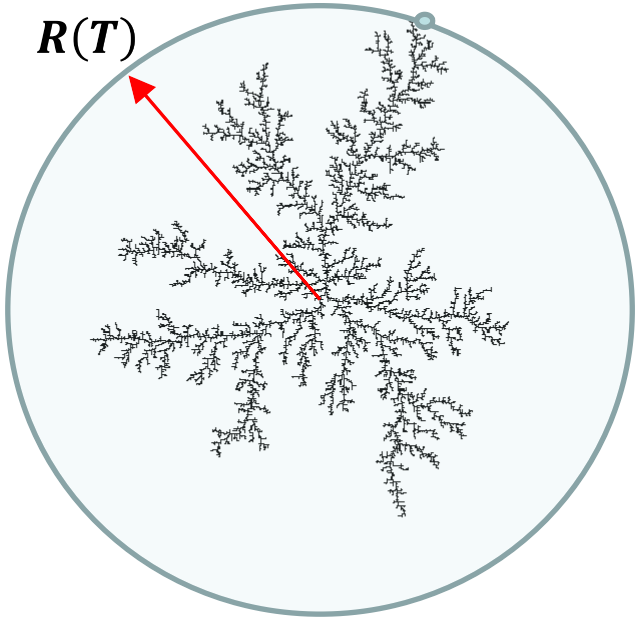


ω inside and outside a snowflake, by D. Marshall

DLA and DBM questions

Denote by $R = R(T)$ the radius of the aggregate of T particles. We define the growth rate:

$$\beta := \lim_{T\to\infty} \frac{\log R(T)}{\log T} \quad \textit{(should do upper/lower)}$$



Clearly $R \lesssim T \lesssim R^2$ so $\frac{1}{2} \leq \beta_* \leq \beta^* \leq 1$

It is widely believed that DLA is an "honest" fractal, with $\beta_* = \beta^*$, and $\boldsymbol{R \asymp T^{\beta}}$,
while $\boldsymbol{\rho := 1/\beta}$ is the dimension: $\boldsymbol{T \approx R^{\rho}}$, $1 \leq \rho \leq 2$.

- **What is the value of ρ for DLA and η-DBM?**
- **Is it strictly between 1 and 2 ?**
- **Is DLA or η-DBM a self-similar fractal ?**

Celebrated theorem by Harry Kesten: for DLA $\rho \geq 3/2$

Proof by Kesten: for DLA $\rho \geq 3/2$

For the radius $R(T)$ of a T-particle cluster to increase, we need to attach a particle at furthermost point, say k, the probability is

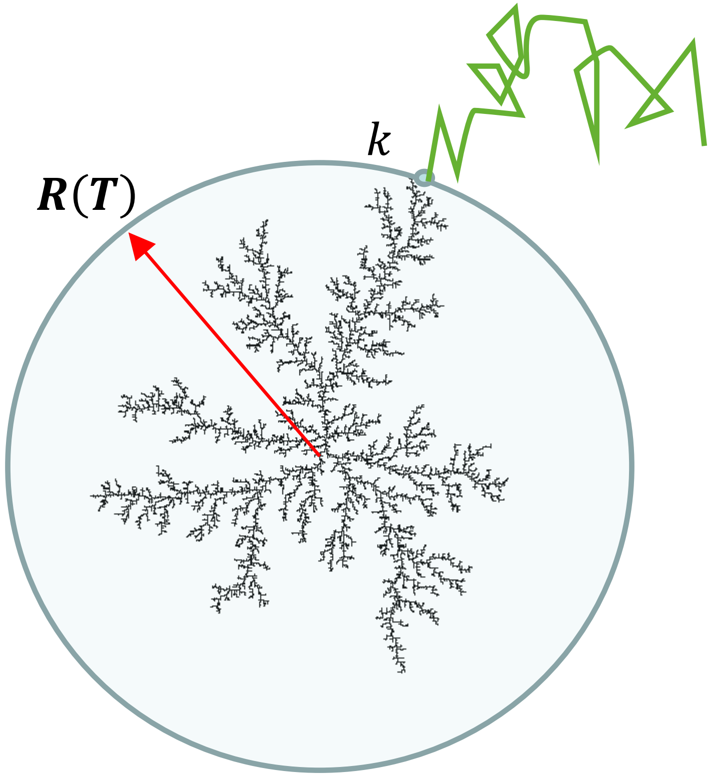
$$\mathbb{P}_{\boldsymbol{k}} = \boldsymbol{\omega_k} \leq \mathbf{sup}_{\boldsymbol{j}} \boldsymbol{\omega_j} =: \boldsymbol{R^{-\sigma}}$$

with $\boldsymbol{\sigma} \geq \mathbf{1/2}$ by Beurling.

Hence $\quad \mathbb{E}\, \boldsymbol{\partial_T R(T)} \lesssim \boldsymbol{R(T)^{-\sigma}} \quad (\heartsuit)$

Informally: assume $R(T) \asymp T^{\beta}$, then $(\heartsuit)$ translates into $T^{\beta-1} \asymp \partial_T T^{\beta} \lesssim T^{-\beta\sigma}$,

so $\beta - 1 \leq -\beta\sigma$, therefore $\boldsymbol{\beta} \leq \frac{\mathbf{1}}{\mathbf{1}+\boldsymbol{\sigma}}$ and for dimension $\beta^{-1} = \boldsymbol{\rho} \geq \mathbf{1} + \boldsymbol{\sigma}\ (\spadesuit)$, so $\boldsymbol{\rho} \geq \frac{\mathbf{3}}{\mathbf{2}}$ $\blacksquare$



$\boldsymbol{R \approx T^{\beta}, 1/2 \leq \beta \leq 1}$
$\boldsymbol{T \approx R^{\rho}, 1 \leq \rho \leq 2}$

Potentially, there are several furthermost points. This is overcome by summing over all possible ways to reach a given radius (count self-avoiding walks!)

ω : fjords and spikes

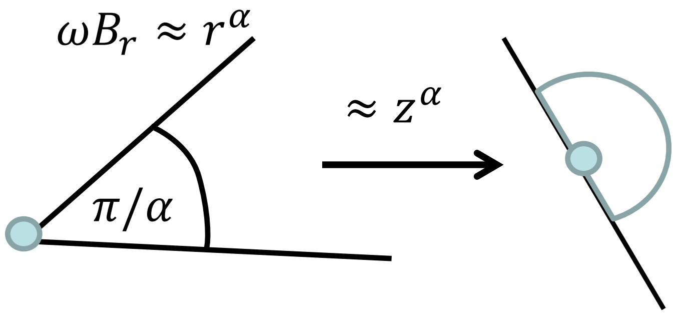
Many important properties of $\boldsymbol{\omega}$ can be written in terms of multifractal spectrum.

Let $\mathcal{F}_\alpha := \{z: \omega B(z,r) \approx r^\alpha\}$

To be precise, one has to take lower/upper limits while $r \to 0$.

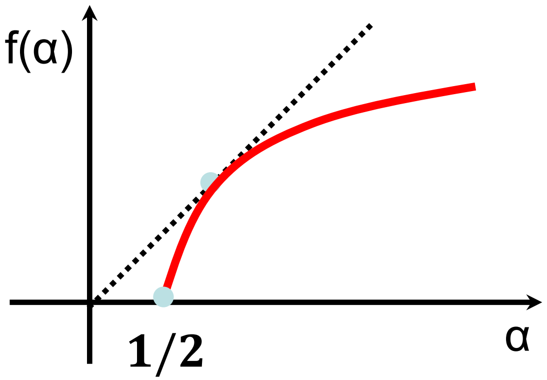
Def ***multifractal spectrum*** is given by

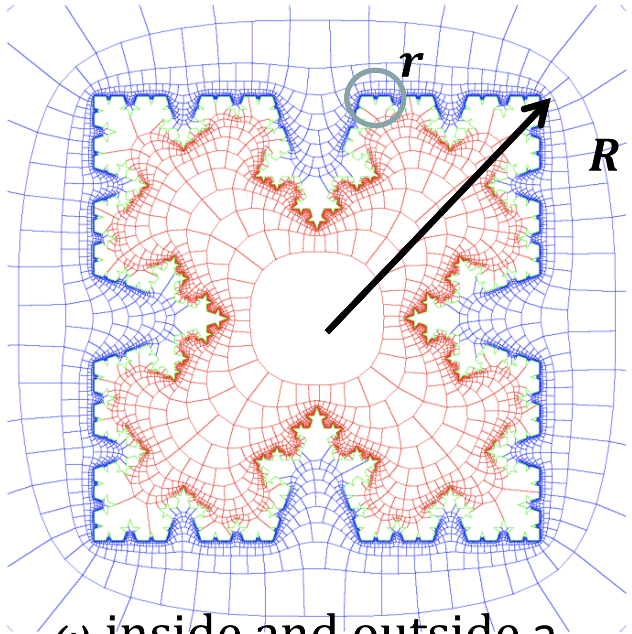
$$f(\alpha) := \mathrm{HDim}(\mathcal{F}_\alpha)$$



Important conjecture by Brennan-Carleson-Jones-Krätzer-Makarov states that

$$\sup_{\text{All domains}} f(\alpha) = 2 - 1/\alpha$$





ω inside and outside a snowflake, by D. Marshall

ω : fjords and spikes

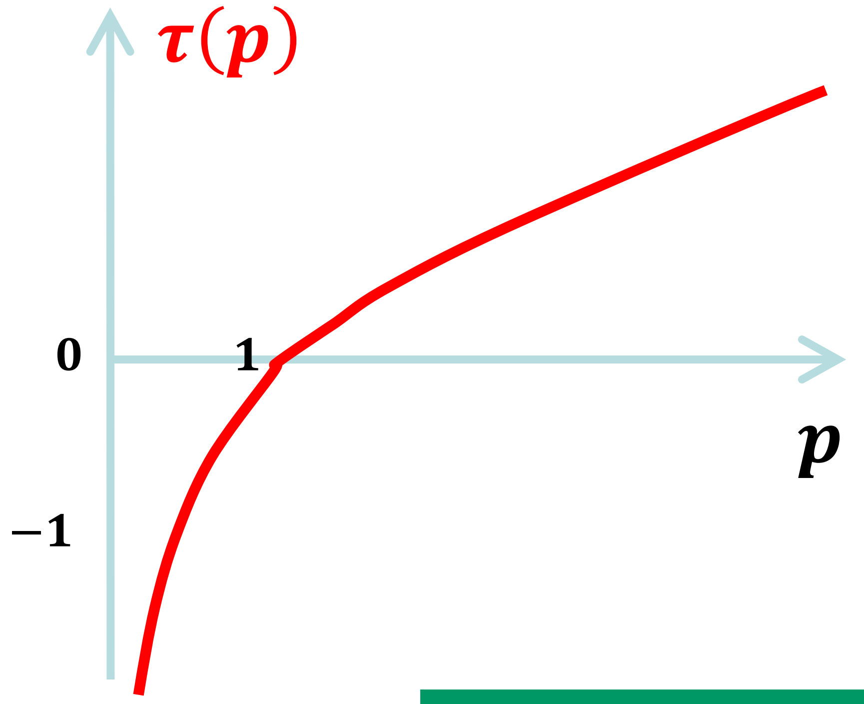
A different spectrum is given (at scale $r = 1$) by

$$\boldsymbol{\tau(p) := -\log_R Z(p), \;\; Z(p) := \sum_j \omega_j^p}$$

Note: related to $f(\alpha)$ by Legendre transform.

Properties:

- Concave



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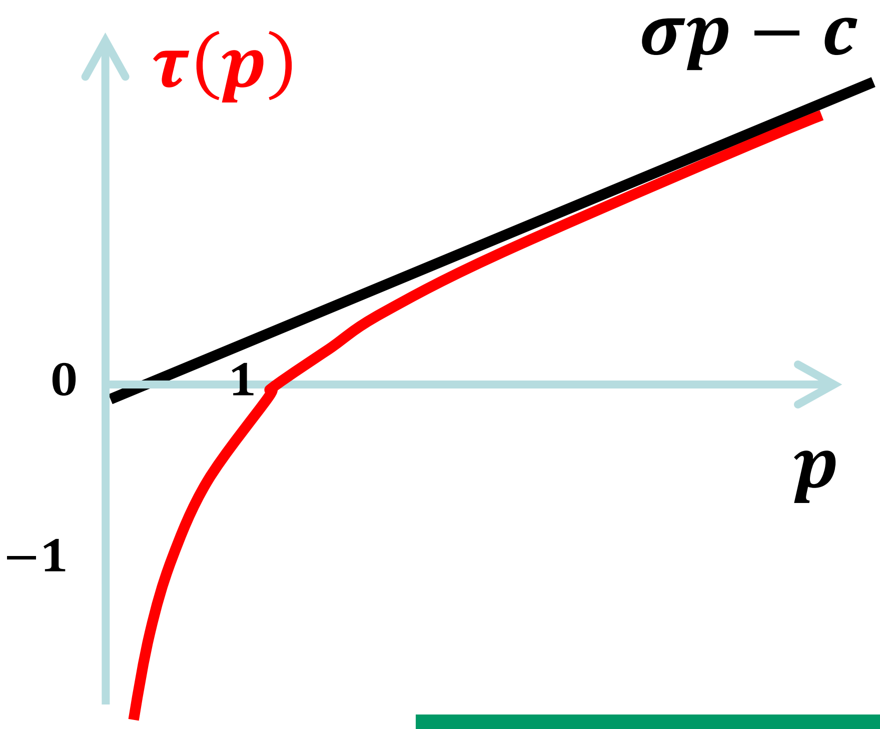
Properties:

- Concave
- Asymptotically $\boldsymbol{\sigma p - c}$ with
  $\mathbf{sup}\, \boldsymbol{\omega_j = R^{-\sigma}}, c \in [0,1]$
  Also $\boldsymbol{\sigma \geq \tau(a)/a}$, easily follows
  from $\mathrm{R}^{-a\sigma} \leq \sum_j \omega_j^a \asymp R^{-\tau(a)}$



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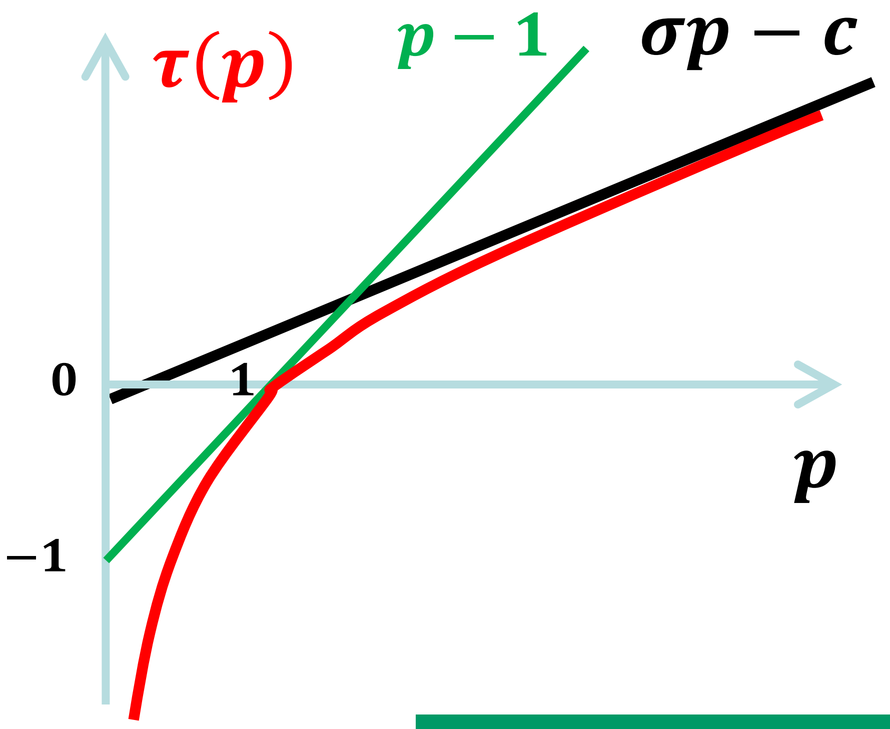
  Also $\boldsymbol{\sigma \geq \tau(a)/a}$, easily follows
  from $\mathrm{R}^{-a\sigma} \leq \sum_j \omega_j^a \asymp R^{-\tau(a)}$
- Makarov:
  $\boldsymbol{\tau(1) = 0,\ \tau'(1) = 1}$
  $\boldsymbol{\tau(p) \leq p - 1}$



Our theorem: for DBM $\rho \geq 2 - \eta/2$

Theorem [Losev-S] For Dielectric Breakdown Model with parameter $0 \leq \eta \leq 2$, dimension satisfies $\rho_{DBM} \geq 2 - \eta/2$

Two proofs: 1) variation of Kesten + discrete version of Makarov;
2) multifractal, cf. Halsey and Lawler.

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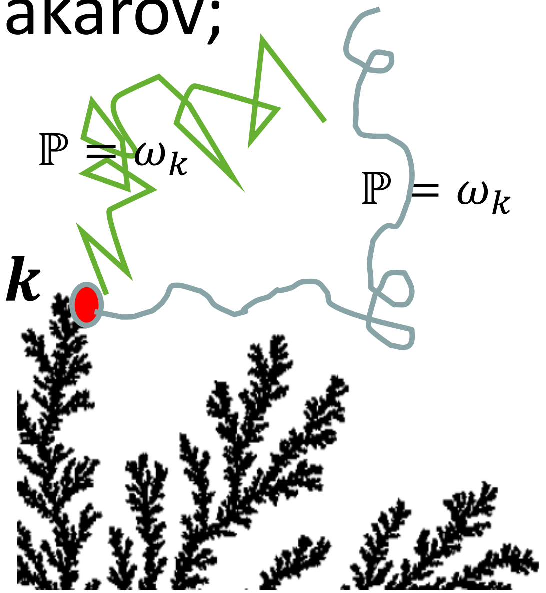
Proof 2) for DLA: Assume for simplicity

$$R \asymp T^{\beta} \text{ and } \sum_j \omega_j^p \asymp R^{-\tau(p)}$$



$\mathbb{P}\text{excursion} \asymp$
$\asymp \mathbb{P}\text{in} \times \mathbb{P}\text{out}$
$\asymp \omega_k \times \omega_k$

Capacity comparable to log-radius: $\mathbf{Cap}(A) \asymp \log R$ and its increment when a particle lands at a site k, satisfies

$$\mathbf{Cap}(A_{T+1}) - \mathbf{Cap}(A_T) \asymp \omega_k^2$$

- *Can also see by looking at changes in partition function of RW*
- *Increments of* $\mathbf{Cap}(A)$ *and* $\log R$ *are different, but time averages are comparable!*

Our theorem: for DBM $\rho \geq 2 - \eta/2$

Increment of capacity when a particle lands at a site $\boldsymbol{k}$ is $\asymp \boldsymbol{\omega_k^2}$, and probability to land there is $\boldsymbol{\omega_k}$, so (on average over $\boldsymbol{T}$)

$$\mathbb{E}\,\partial_T \log R(T) \asymp \mathbb{E}\,\partial_T\,\mathrm{Cap}(A_T) \asymp \sum_k \omega_k^2 \times \omega_k = \sum_k \omega_k^3 = Z(3) \quad (\Diamond)$$

Assuming $\boldsymbol{R(T) \asymp T^{1/\rho}}$, and $\boldsymbol{Z(3) \asymp R^{-\tau(3)}}$ we conclude

$$R^{-\rho} \asymp T^{-1} \asymp \partial_T \log T^{1/\rho} \asymp Z(3) \asymp R^{-\tau(3)}$$

Hence $\boldsymbol{\rho = \tau(3)}$ (observed by Halsey), recall also $\boldsymbol{\sigma \geq \tau(a)/a}$

We do not use Beurling's $\boldsymbol{\sigma \geq 1/2}$, can deduce it from the DLA properties!

Start with Kesten's $\boldsymbol{\rho \geq 1 + \sigma}$ $(\spadesuit)$ and write

$$\rho \geq 1 + \sigma \geq 1 + \tau(3)/3 = 1 + \rho/3$$

Hence $\rho(1 - 1/3) \geq 1$ and $\boldsymbol{\rho \geq 3/2}$ $\blacksquare$

What's next? Is DLA/DBM a fractal?

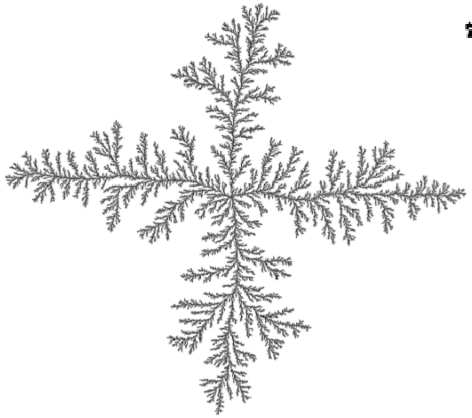
Clearly, DLA has *subsequential* scaling limits by precompactness:
Consider it as a random set (or a tree or a mass distribution, or …), rescale to size 1, tend $T \to \infty$.

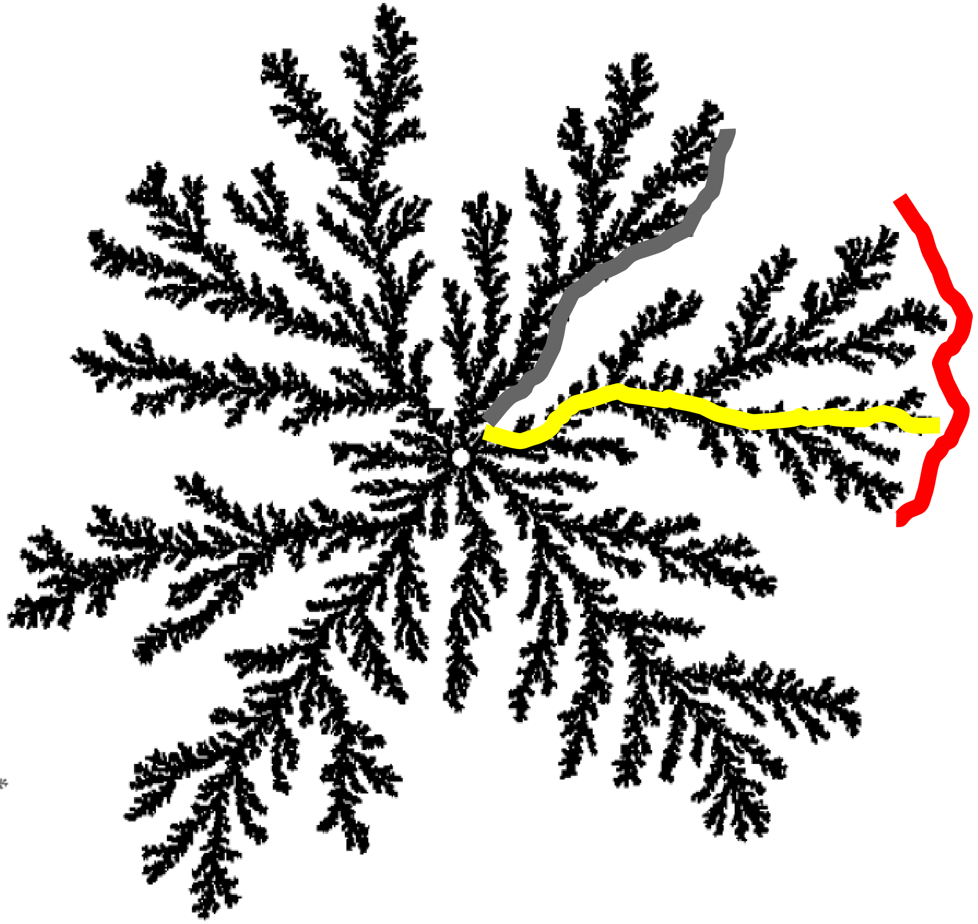
But it might turn out a small perturbation of a disk…

Still one can study

- Branches
- Boundaries of branches
- Frontiers

Also the difference with the lattice DLA remains open





Assume similarity of one scale
⇒ either ρ=2 or spikes dominate

An unrigorous argument

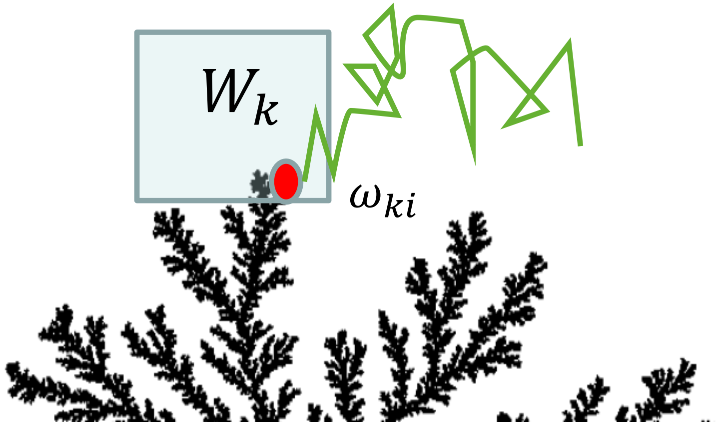
How long does it take to "fill" square W_k?

We need to accumulate capacity $\approx W_k^2$ (♠)

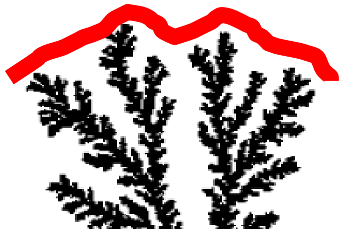
Particle landing at site ik adds capacity $\approx \omega_{ik}^2$

The expected increase rescaled by the goal of W_k^2

$$\approx \sum_i \omega_{ik}^3 W_k^{-2} = \sum_i \left(\frac{\omega_{ik}}{W_k}\right)^3 W_k \asymp r^{-\rho} W_k$$



This is similar to scale 1, but we need r^ρ / W_k steps, and more exposed squares (larger W_k) fill much faster, obscuring other regions.


A

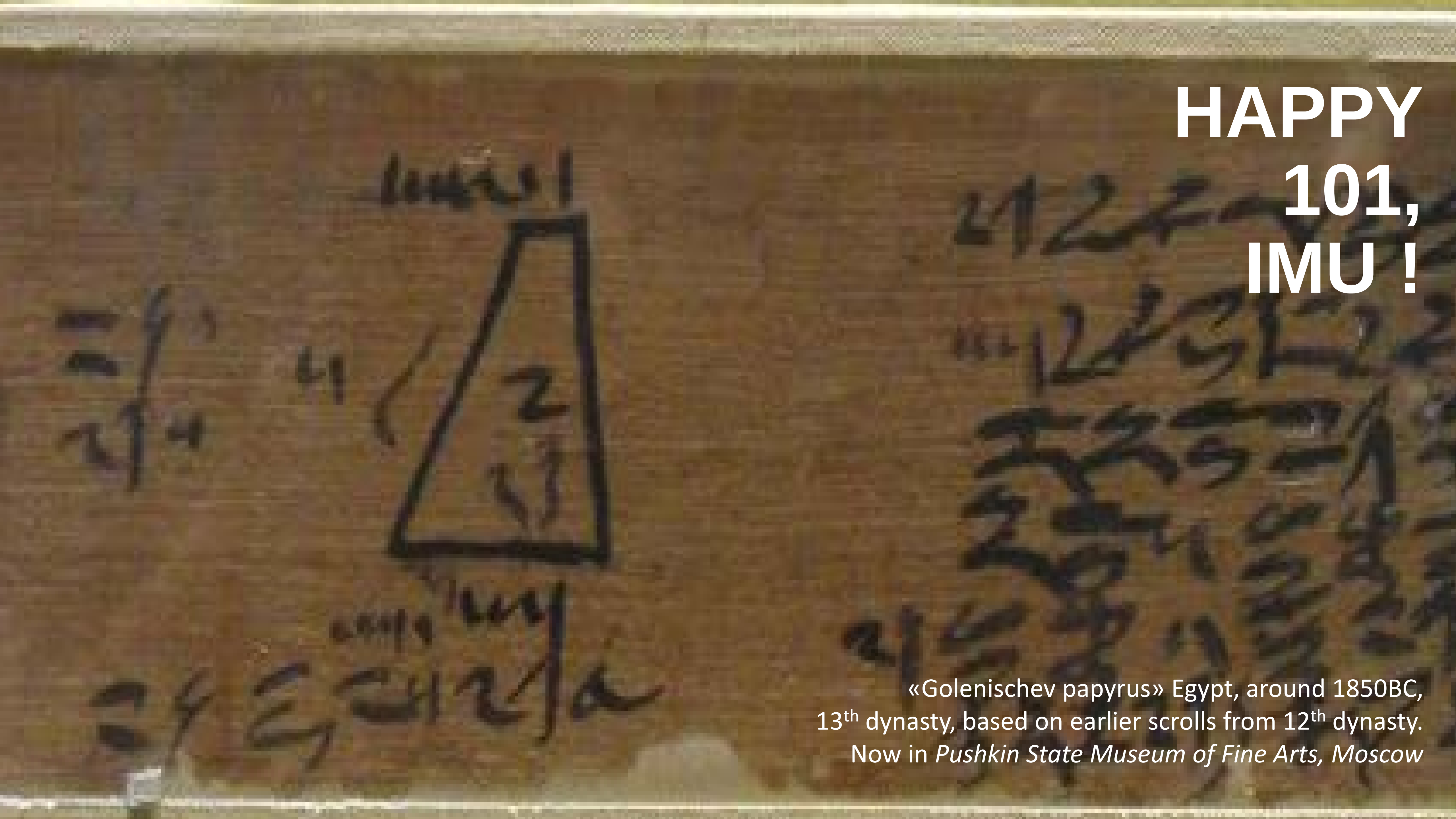
Two possibilities:

A. Only Makarov-flat boundary: $W \lesssim 1/r$, uniform growth

B. Growth dominated by spikes


B

HAPPY BIRTHDAY,
IMU !



HAPPY 101, IMU !

«Golenischev papyrus» Egypt, around 1850BC,
13th dynasty, based on earlier scrolls from 12th dynasty.
Now in *Pushkin State Museum of Fine Arts, Moscow*