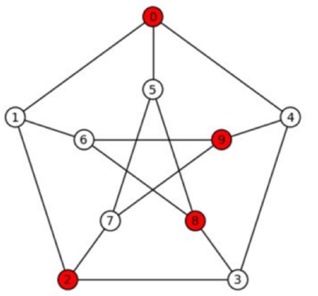
Hardness of Approximation

From the PCP Theorem to the 2-to-2 Games Theorem

$x_1 - x_7 = 11 \pmod{17}$
$x_2 - x_3 = 13 \pmod{17}$
...
....
$x_7 - x_9 = 15 \pmod{17}$



Subhash Khot

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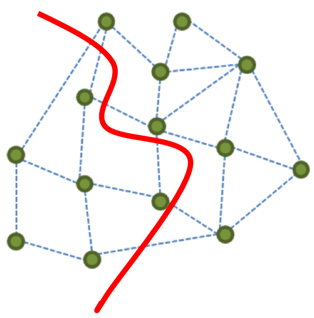
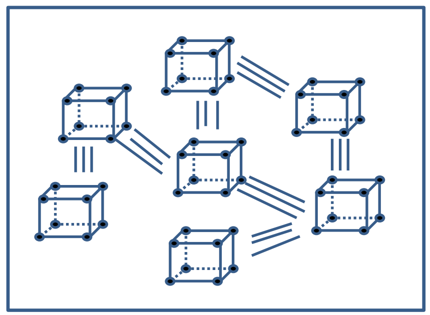
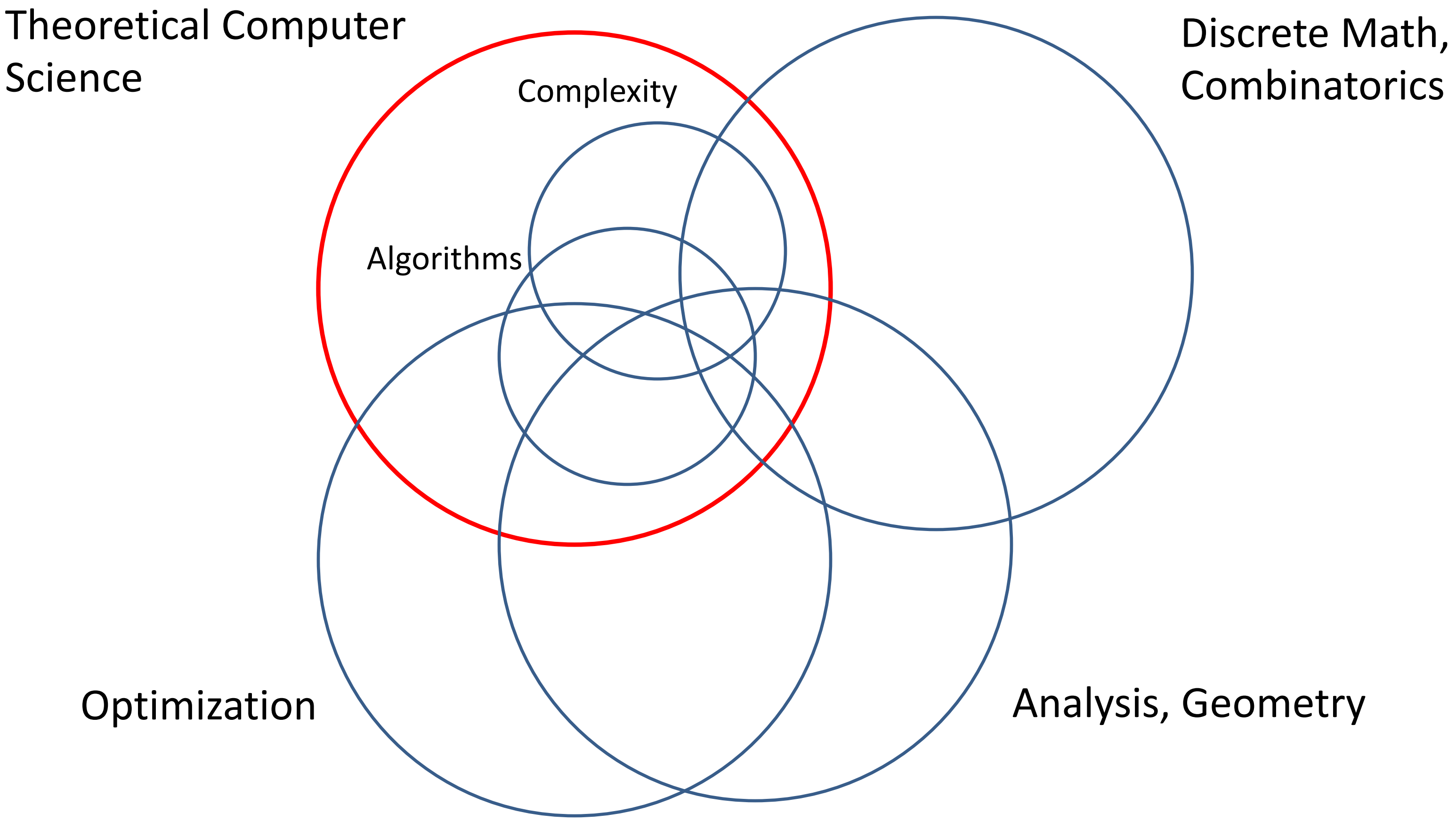
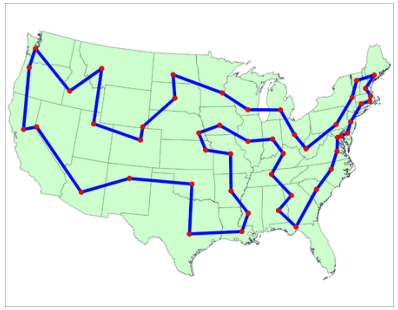




Where it Fits



NP-hard Problems



Traveling Salesperson

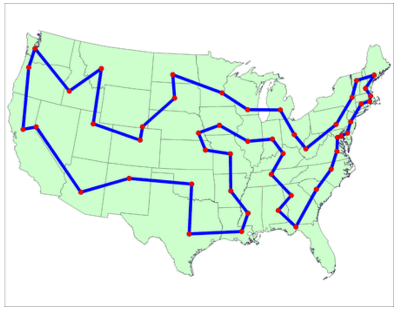
$$(x \text{ OR } y \text{ OR } z) \text{ AND } (x \text{ OR } \bar{y} \text{ OR } z) \text{ AND}$$
$$(x \text{ OR } y \text{ OR } \bar{z}) \text{ AND } (x \text{ OR } \bar{y} \text{ OR } \bar{z}) \text{ AND}$$
$$(\bar{x} \text{ OR } y \text{ OR } z) \text{ AND } (\bar{x} \text{ OR } \bar{y} \text{ OR } \bar{z})$$

3-SAT

- Everywhere, look under your chair! All equivalent to each other.
- $P \neq NP \equiv$ There is no fast (polynomial time) algorithm.
- Can we compute approximate solutions fast? (practice, theory, math).

NP-hard Problems

How well can we approximate?



(x OR y OR z) AND (x OR $\bar{y}$ OR z) AND

(x OR y OR $\bar{z}$) AND (x OR $\bar{y}$ OR $\bar{z}$) AND

($\bar{x}$ OR y OR z) AND ($\bar{x}$ OR $\bar{y}$ OR $\bar{z}$)

Traveling Salesperson

3-SAT

Within 1%
[Arora, Mitchell 98]

$\frac{7}{8}$ but not better
[Håstad 96]

- Focus of this talk: Hardness of approximation
- Amazing progress so far. Many challenges remain.
- Exact versus Approximate: a different ballgame.

Hardness of Approximation: Historically ……

1970	NP-hardness	[Cook, Karp, Levin]
1990	PCP Theorem	[Arora, Babai, Feige, Fortnow, Goldwasser, Lovasz] [Karloff, Lund, Motwani, Nisan, Safra, Shamir, Sudan, Szegedy]
1995- 2015	Multi-Prover Games, Boolean Function Analysis	[Bellare, Chan, Dinur, Feige, Goldreich, Guruswami, Håstad, K] [Kindler, Moshkovitz, Mossel, O'Donnell, Raz, Regev] [Raghavendra, Safra, Samorodnitsky, Sudan, Steurer, Trevisan]
2018	2-to-2 Games Theorem	[Dinur, K, Kindler, Minzer, Safra]
??	Unique Games Conjecture	[K]
??	Small Set Expansion Conjecture	[Raghavendra, Steurer]

Overview of the Talk

- Independent Set Problem

  Checking a proof without looking at it.

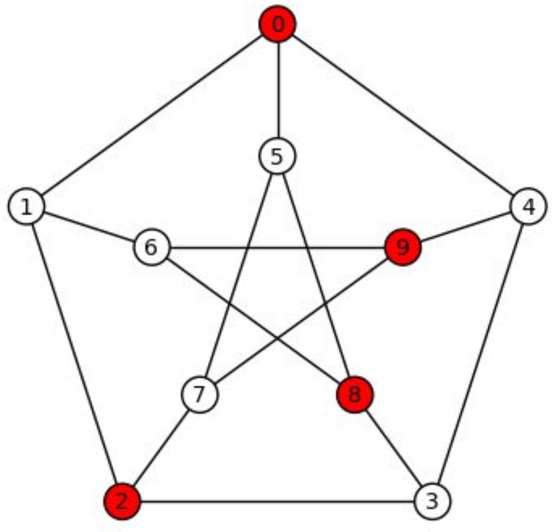
- Maximum Cut Problem

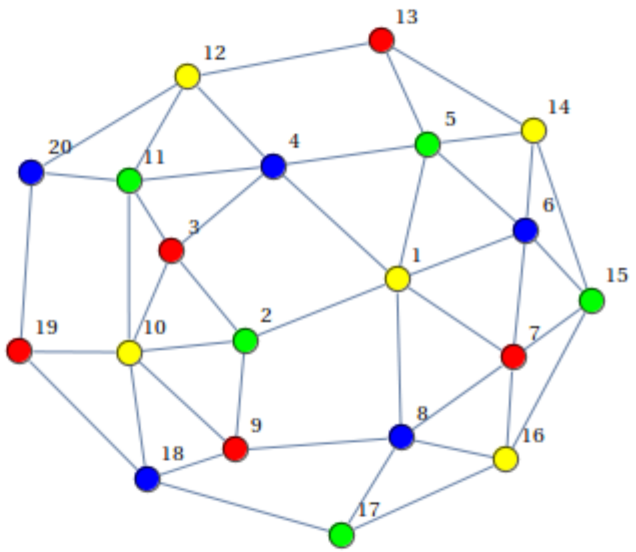
  Majority vote, isoperimetry, spherical cubes.

- 2-to-2 Game Problem

  NP-hard despite all the evidence otherwise.

Independent Set Problem





Problem

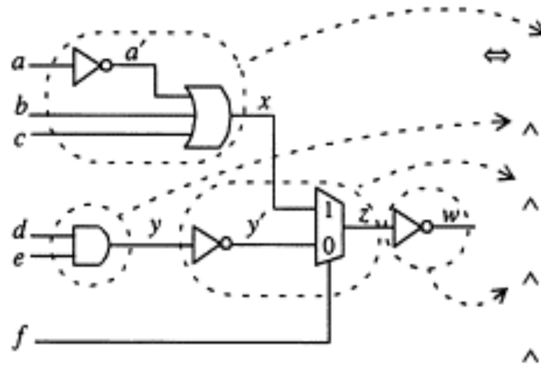
Given: A graph with N vertices that contains an independent set of size $\frac{N}{4}$

Goal: Find a large independent set.

2018 2-to-2 Games Theorem:

NP-hard to find independent set of size 0.001 N.

Checking Proofs of Satisfiability



(a)

$$\Leftrightarrow (a \vee x) \wedge (\neg b \vee x) \wedge (\neg c \vee x) \wedge (\neg a \vee b \vee c \vee \neg x)$$
$$\wedge (d \vee \neg y) \wedge (e \vee \neg y) \wedge (\neg d \vee \neg e \vee y)$$
$$\wedge (\neg x \vee \neg f \vee z) \wedge (x \vee \neg f \vee \neg z) \wedge (y \vee f \vee z) \wedge (\neg y \vee f \vee \neg z)$$
$$\wedge (\neg z \vee \neg w) \wedge (z \vee w)$$
$$\wedge\, w$$

(b)

Satisfiability formula: Φ

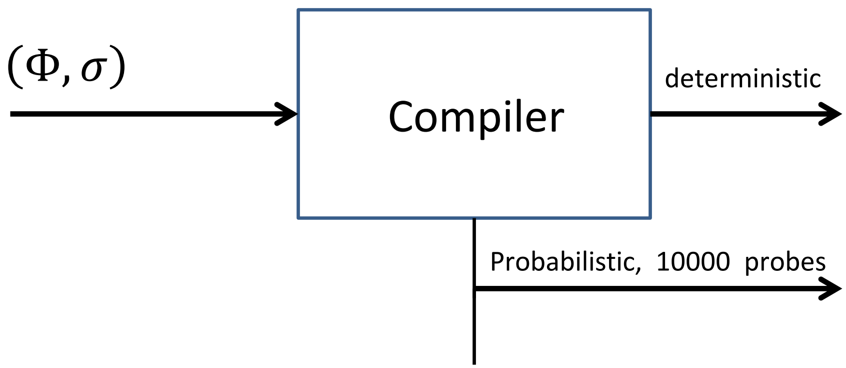
Fate of the word depends on knowing if Φ has a satisfying assignment.
if so, what is it.

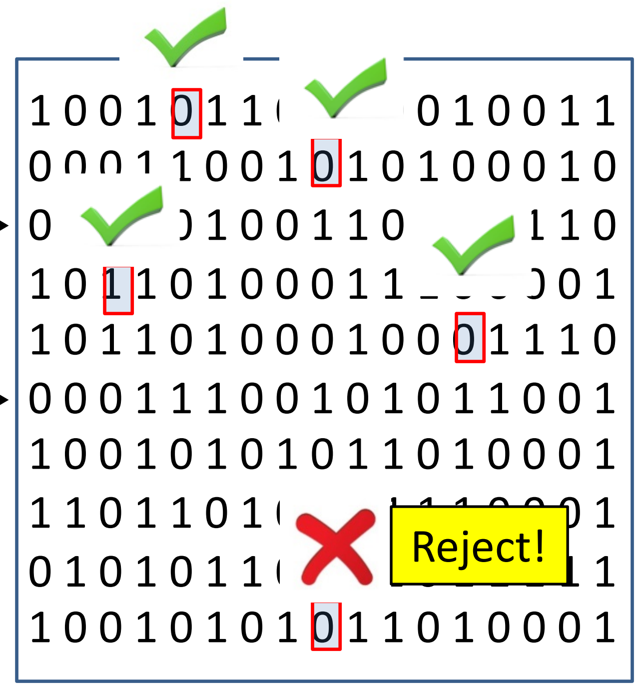
Crackpots claiming to discover satisfying assignment σ.

Can we minimize effort of “talking” to the crackpots and “verifying” their claim?

Checking Proofs without Looking

2018 2-to-2 Games Theorem





Accept if $[a\ b\ c\ d\ e] = [0\ 0\ 1\ 0\ 1]$

- If σ does not satisfy Φ then Pr [accept] < 0.0001

  If Φ is unsatisfiable then Pr [accept] < 0.0001 irrespective of the proof.

- If σ does satisfy Φ then Pr [accept] = 1/4

Overview of the Talk

- Independent Set Problem

  Checking a proof without looking at it.

- Maximum Cut Problem

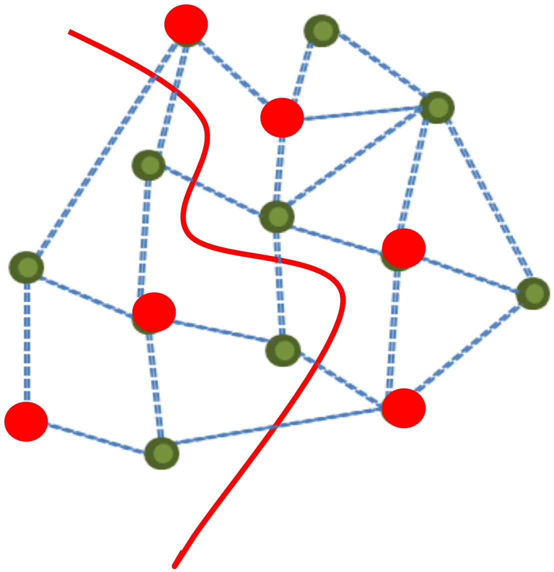
  Majority vote, isoperimetry, spherical cubes.

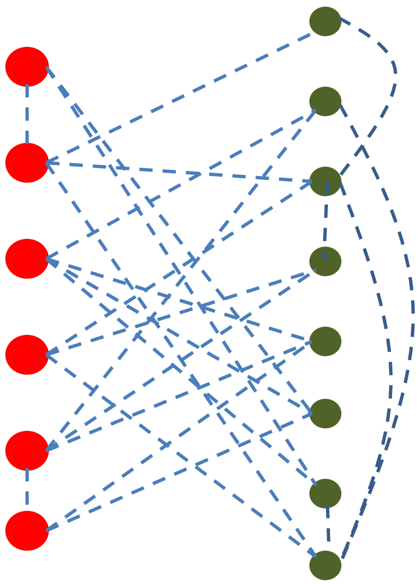
- 2-to-2 Game Problem

  NP-hard despite all the evidence otherwise.

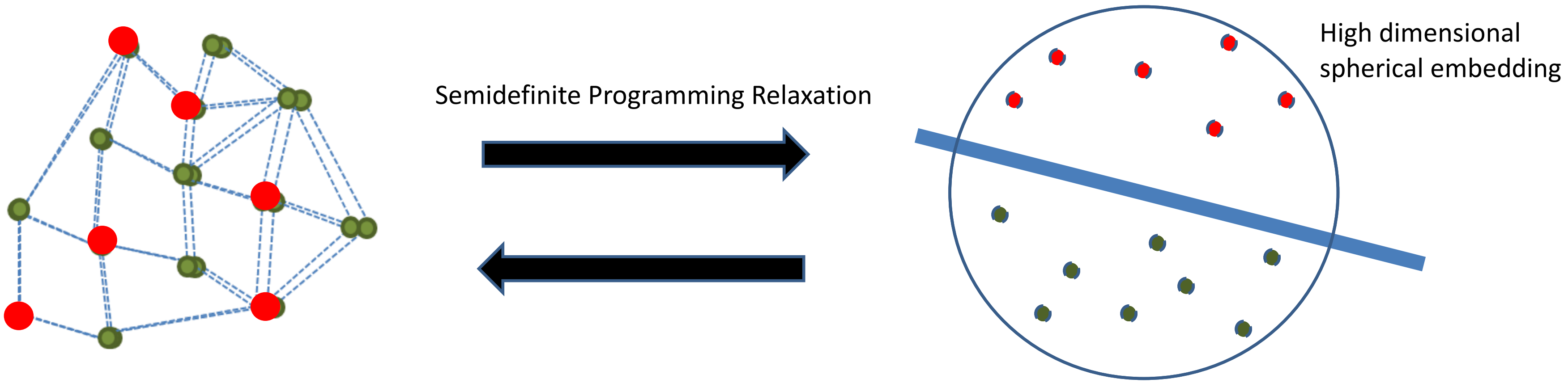
Maximum Cut Problem

Find a cut that maximizes the number of edges cut.





Goemans-Williamson Algorithm '90



- Unique / 2-to-2 Games Conjecture $\Rightarrow$ [Goemans Williamson] is optimal!

  - Majority is Stablest.

  - Isoperimetric problems.

Reduction, Majority Is Stablest Theorem

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….
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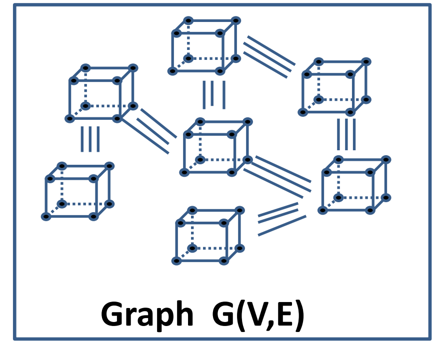
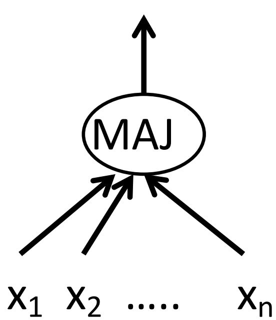
Unique /2-to-2 Game instance

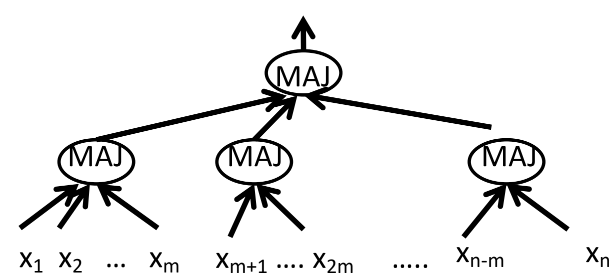
Reduction →

Majority Is Stablest Theorem



- $f: \{-1,1\}^n \rightarrow \{-1,1\}$ ≡ "voting schemes"
- Two candidates $\{-1,1\}$, n voters, votes are i.i.d. $\{-1,1\}$.
- Voting scheme $f: \{-1,1\}^n \rightarrow \{-1,1\}$, "democratic: no voter too influential".
- Majority — Electoral College





Majority Is Stablest Theorem

- Voting scheme f : $\{-1,1\}^n \rightarrow \{-1,1\}$, "democratic", "balanced".
- Which f is most noise-stable?

$$x \leftarrow (x_1, \dots, x_n), \qquad \forall i \;\; y_i \leftarrow \begin{cases} x_i & \text{with probability } 1-\varepsilon \\ -x_i & \text{with probability } \varepsilon \end{cases}$$

$$y \leftarrow (y_1, \dots, y_n).$$

Which f minimizes $\Pr_{x,y}[f(x) \neq f(y)]$?

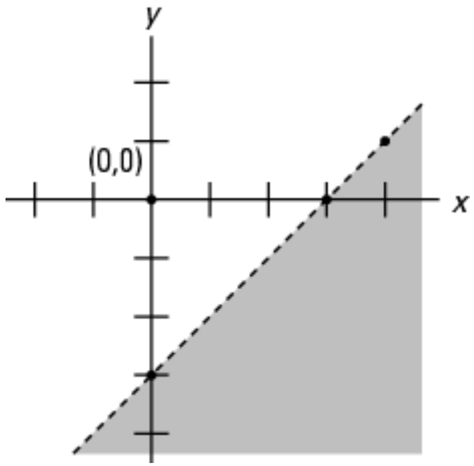
- [Mossel O'Donnell Oleszkiewicz '05] Majority!
- Main idea: Switch from Boolean domain to Gaussian domain.

Isoperimetry

In R^n, which "shape" with a fixed volume has least surface area?

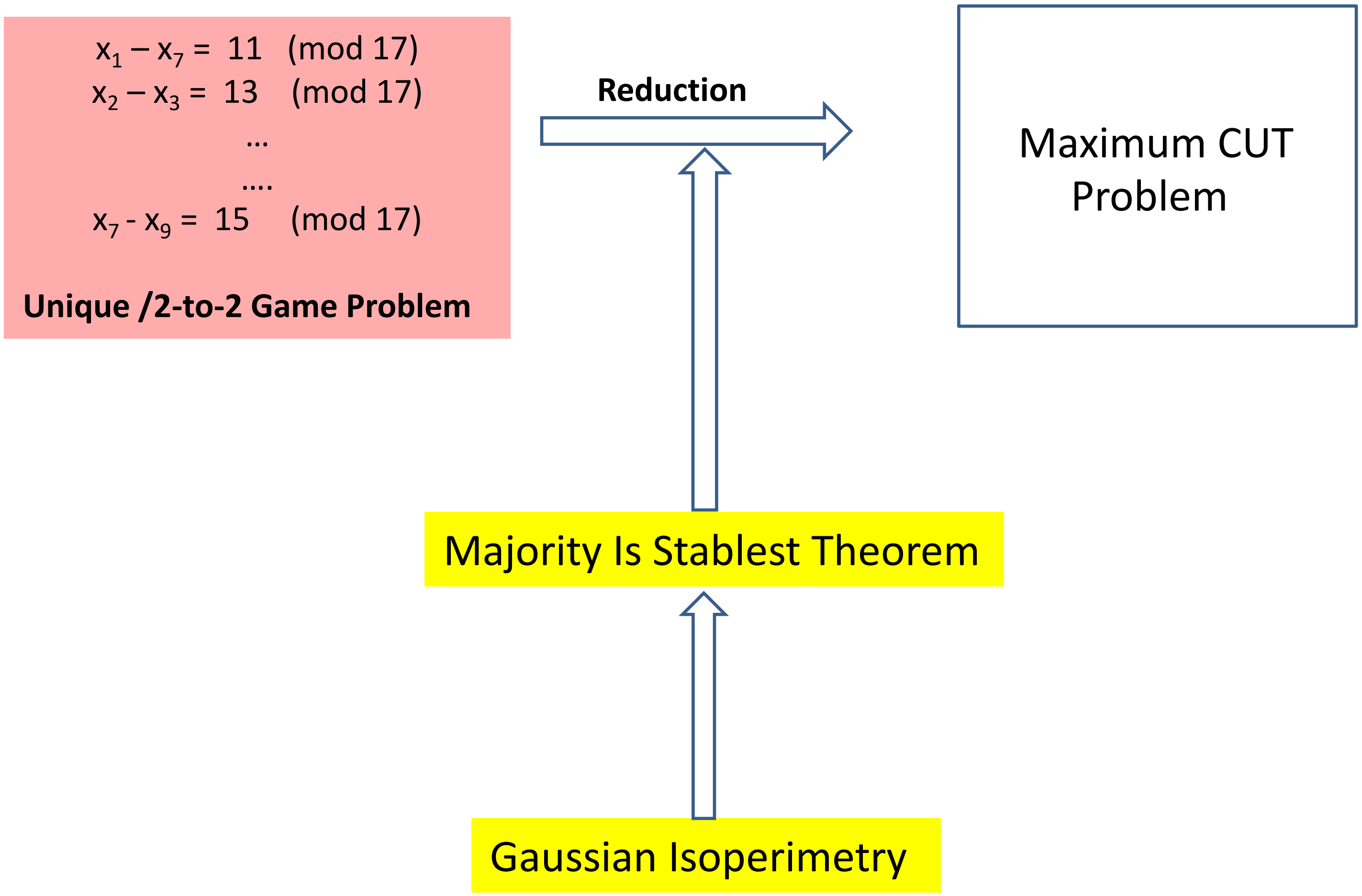
Standard case:

Gaussian case:



[Borell '85]

Punchline



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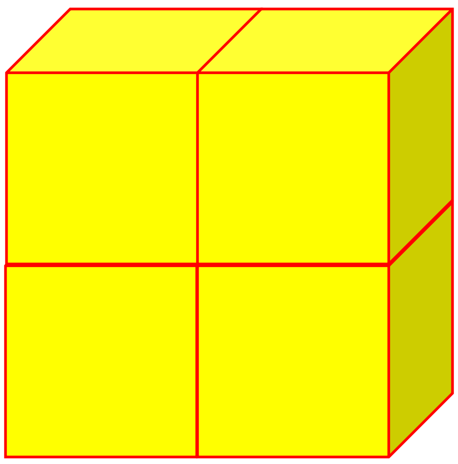
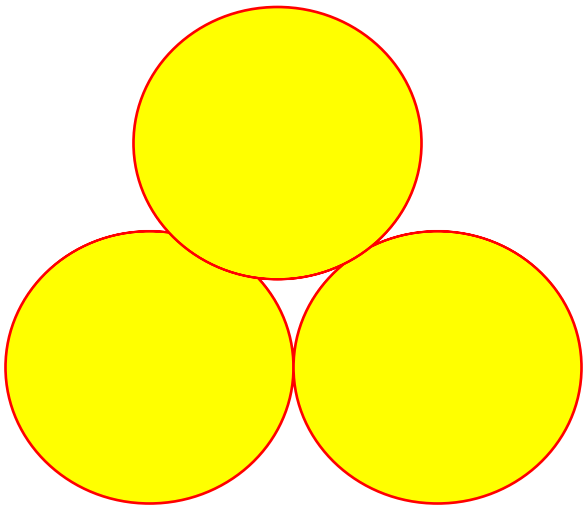
$x_2 - x_3 = 13 \pmod{17}$
…
….
$x_7 - x_9 = 15 \pmod{17}$
Unique /2-to-2 Game Problem
Reduction
Maximum CUT Problem
Majority Is Stablest Theorem
Gaussian Isoperimetry

Spherical Cubes: Hardness Amplification Fails

Problem: Tiling R^n using a "shape" of unit volume and minimum surface area.





[Raz, Kindler O'Donnell Rao Wigderson, Alon Klartag '08]

There exists a tiling shape with unit volume and surface area $O(\sqrt{n})$!

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  NP-hard despite all the evidence otherwise.

Unique Game

$x_1 - x_7 = 5 \pmod{p}$

$x_2 - x_3 = -2 \pmod{p}$

....

$x_i - x_j = c_{ij} \pmod{p}$

.....

$x_7 - x_n = 11 \pmod{p}$

2-to-2 Game

$x_1 - x_7 = 5, -1 \pmod{p}$

$x_2 - x_3 = -2, 0 \pmod{p}$

....

$x_i - x_j = c_{ij}, b_{ij} \pmod{p}$

.....

$x_7 - x_n = 11, 3 \pmod{p}$

Unique Games Conjecture

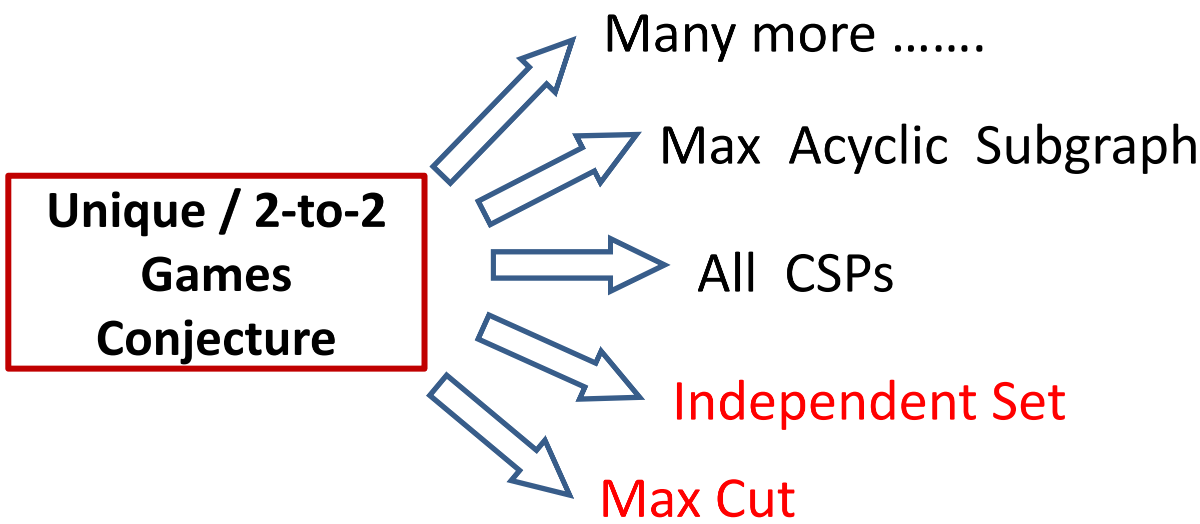
$\forall\ \delta > 0,\ \exists\ p$ such that given a $1-\delta$ satisfiable Unique Game,
it is NP-hard to find a δ satisfying assignment.

2-to-2 Games Theorem

$\forall\ \delta > 0,\ \exists\ p$ such that given a ~~$1-\delta$~~ $\frac{1}{2}$ satisfiable ~~2-to-2 Game~~ Unique Game,
it is NP-hard to find a δ satisfying assignment.

Simplest Hard Problem

Many other problems are hard to approximate.



- Algorithms, Optimization.
- Computational complexity.
- Analysis and Geometry.