New approaches for moduli spaces

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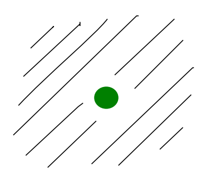
Manifolds

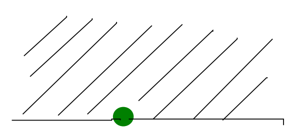
M^d smooth manifold, dimension d

Diff(M) group of diffeomorphisms

BDiff(M) classifying space for smooth M-bundles

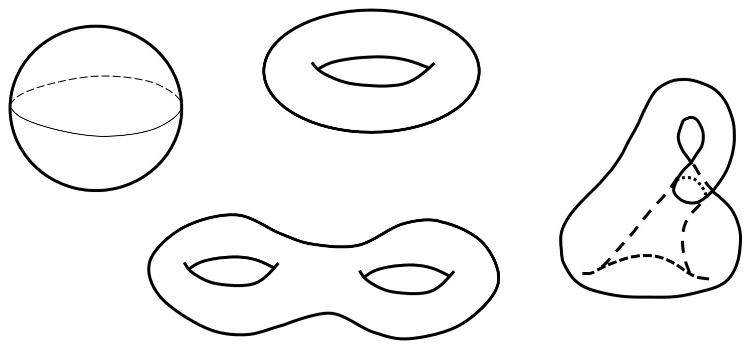
$H^*(B\text{Diff}(M))$ cohomology with coefficients in $\mathbb{Q}$ −
. characteristic classes for M-bundles





$\mathbb{R}^d$ $\qquad$ $\mathbb{R}^d_{\geq 0}$.

Surfaces

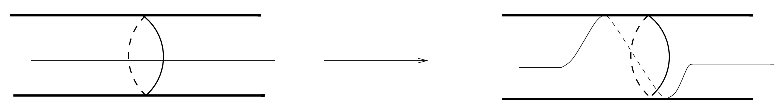


F_g oriented, surface of genus $g \geq 2$

$H^*(B\mathrm{Diff}(F_g)) = H^*(B\Gamma_{g,1}) = H^*(\mathcal{M}_g)$

. $\mathcal{M}_g$ moduli space of Riemann surfaces

Dehn twist:



Mumford conjecture

$F_{g,1} = D^2 \sharp_g (S^1 \times S^1)$

Harer '85: For $g \gg *$,
$H_*(B\mathsf{Diff}_\partial(F_{g,1})) = H_*(B\Gamma_{g,1})$ is independent of g

Madsen-Weiss '07:

$$\lim_{g\to\infty} H_*(B\mathsf{Diff}_\partial(F_g,1)) = H_*(\Omega^\infty \mathbf{MTSO}(2))$$
$$\simeq \mathbb{Q}[\kappa_i \,|\, i \in \mathbb{N}_{>0}]$$

Generalisation to higher dimensions

$W_{g,1} = D^d \sharp_g (S^n \times S^n), \quad d = 2n \geq 6$

Galatius – Randal-Williams '18:

$H_*(B\mathrm{Diff}_\partial(W_{g,1}))$ is independent of g for $g \gg 0$

Galatius – Randal-Williams '14:

$$\lim_{g \to \infty} H_*(B\mathrm{Diff}_\partial(W_g, 1)) = H_*(\Omega^\infty \mathbf{MTSO}(2n) < n >)$$
$$\simeq \mathbb{Q}[\kappa_c \,|\, c \in \mathcal{B}]$$

where $\mathcal{B} =$ set of monomials in $e, p_{n-1}, \ldots, p_{\lceil \frac{n+1}{4} \rceil}$

Weiss fibre sequence

$$D^{d-1} \subset \partial M$$

$$B\mathsf{Diff}_\partial(D^d) \longrightarrow B\mathsf{Diff}_\partial(M) \longrightarrow B\mathsf{Emb}^{\cong}_{\partial/2}(M)$$

Pontryagin-Weiss classes

$O(d) \to \mathsf{Top}(d) = \mathsf{Homeo}(\mathbb{R}^d)$
induces $BO(d) \to B\mathsf{Top}(d)$ and $BO \to B\mathsf{Top}$

$H^*(BO) = H^*(B\mathsf{Top}) = \mathbb{Q}[p_i \,|\, i \in \mathbb{N}_{>0}]$

Well-known relations:

$$p_n = e^2 \qquad \in H^{4n}(BSO(2n))$$
$$p_{n+k} = 0 \qquad \in H^{4n+k}(BO(2n)) \quad k > 0$$

Weiss '15: These relations often fail in $H^*(B\mathsf{Top}(2n))$

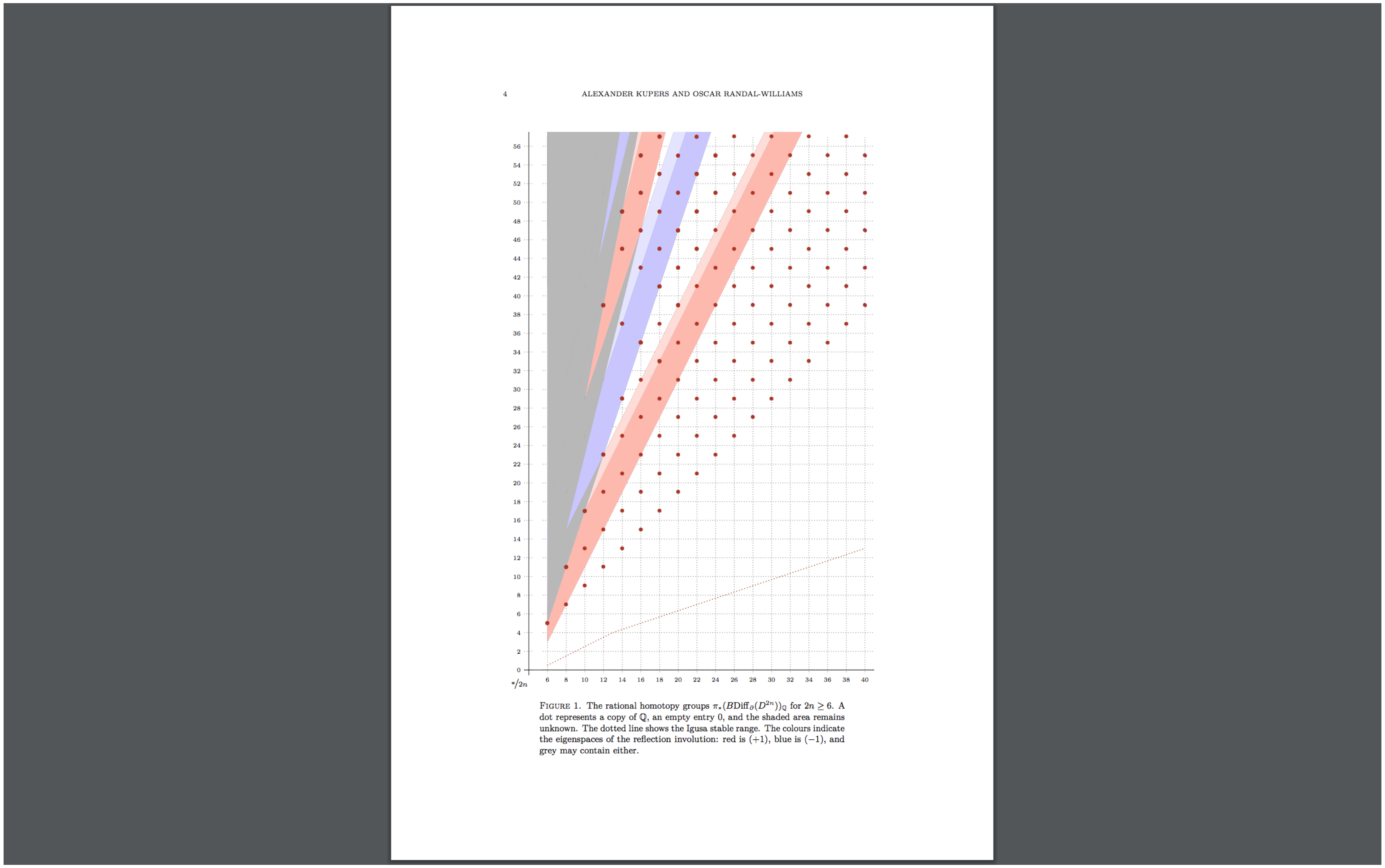
Disks and spheres

$\mathrm{Diff}(S^1) = O(2)$		
$\mathrm{Diff}(S^2) = O(3)$	$\mathrm{Diff}_\partial(D^2) = *$	**Smale '58**
$\mathrm{Diff}(S^3) = O(4)$	$\mathrm{Diff}_\partial(D^3) = *$	**Hatcher '83**
$\mathrm{Diff}(S^4) \neq O(5)$	$\mathrm{Diff}_\partial(D^4) \neq *$	**Watanabe '18**

$\mathrm{Diff}(S^d) = O(d+1) \times \mathrm{Diff}_\partial(D^d)$

For $d \geq 5$,
$\pi_0 \, \mathrm{Diff}_\partial(D^d) \simeq \Theta_{d+1}$ exotic $d+1$-spheres

Many new results for $\pi_*(\mathrm{Diff}_\partial(D^d)) \otimes \mathbb{Q}$ for $d \geq 6$



Kupers – Randal-Williams ’20: $\pi_*(B\mathrm{Diff}_\partial(D^d)) \otimes \mathbb{Q}$ for d **even**

Krannich – Randal-Williams ’21: $\pi_*(B\mathrm{Diff}_\partial(D^d)) \otimes \mathbb{Q}$ for d **odd**

Two stabilisations

$W^d \rightsquigarrow W \times I \rightsquigarrow \cdots \rightsquigarrow W \times I^k$

$I = [0,1]$

Igusa '88: stability theorem ($* <\sim d/3$ and $k \gg 0$)

$$\pi_*(B\mathsf{Diff}_\partial(D^{2n})) \otimes \mathbb{Q} = 0$$

$$\pi_*(B\mathsf{Diff}_\partial(D^{2n+1})) \otimes \mathbb{Q} = K_{*+1}(\mathbb{Z}) \otimes \mathbb{Q}$$

$W^d \rightsquigarrow W\#Q \rightsquigarrow \cdots \rightsquigarrow W\#_g Q$

$Q = S^n \times S^n$

Galatius − Randal-Williams '18: stability ($* <\sim 2g/3$)